

SQUARES AND NARROW SYSTEMS

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ABSTRACT. A narrow system is a combinatorial object introduced by Magidor and Shelah in connection with work on the tree property at successors of singular cardinals. In analogy to the tree property, a cardinal κ satisfies the *narrow system property* if every narrow system of height κ has a cofinal branch. In this paper, we study connections between the narrow system property, square principles, and forcing axioms. We prove, assuming large cardinals, both that it is consistent that $\aleph_{\omega+1}$ satisfies the narrow system property and $\square_{\aleph_{\omega}, < \aleph_{\omega}}$ holds and that it is consistent that every regular cardinal satisfies the narrow system property. We introduce natural strengthenings of classical square principles and show how they can be used to produce narrow systems with no cofinal branch. Finally, we show that the Proper Forcing Axiom implies that every narrow system of countable width has a cofinal branch but is consistent with the existence of a narrow system of width ω_1 with no cofinal branch.

1. INTRODUCTION

The question as to when certain large cardinal properties can hold at non-inaccessible cardinals has been of considerable interest in modern set theory. Of particular interest are successors of singular cardinals (particularly $\aleph_{\omega+1}$), at which these properties are typically more difficult and require larger cardinals to attain than at successors of regular cardinals. One of the large cardinal properties that has received a great deal of attention is the tree property. In [9], Magidor and Shelah prove that the tree property holds at the successor of a singular limit of strongly compact cardinals and that, assuming large cardinals (roughly a huge cardinal with ω -many supercompact cardinals above it), it is consistent that the tree property holds at $\aleph_{\omega+1}$. In the same paper, they introduce the notion of a narrow system, which has proved to be a valuable tool in the analysis of the tree property at successors of singular cardinals and is the primary subject of this paper. In [11], Sinapova reduces the large cardinals needed to obtain the tree property at $\aleph_{\omega+1}$ by forcing it from ω -many supercompact cardinals. In [10], Neeman shows the consistency of the tree property holding simultaneously at $\aleph_{\omega+1}$ and \aleph_n for all $2 \leq n < \omega$ and, in the process, demonstrates a different method for forcing the tree property at $\aleph_{\omega+1}$ from ω -many supercompact cardinals.

The forcing constructions employed by Magidor and Shelah, Sinapova, and Neeman are all distinct, but, in all known models in the which the tree property holds at the successor of a singular cardinal, μ , the verification of the tree property follows the same general two-step pattern. In the first step, it is argued that every μ^+ -tree admits a narrow system of height μ^+ (a precise definition of this will be given later). In the second step, it is argued that every narrow system of height μ^+ has a cofinal

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branch. With this in mind and with an eye towards getting a better understanding of matters surrounding the tree property, we focus our attention here on these two steps, and in particular on the latter, taken individually.

The general structure of the paper is as follows. In Section 2, we introduce narrow systems and prove some basic facts about them. In Section 3, we recall some combinatorial and forcing notions that will be useful throughout the paper. In Section 4, we present branch preservation lemmas for narrow systems. These are slight improvements on a similar lemma of Sinapova from [11]. In Section 5, we present some forcing constructions related to the narrow system property. In particular, starting from large cardinals, we obtain a model in which every narrow system has a cofinal branch, a model in which the narrow system property at $\aleph_{\omega+1}$ and $\square_{\aleph_\omega, < \aleph_\omega}$ both hold, and a model in which there is an inaccessible, non-weakly compact λ such that the narrow system property holds at λ . In Section 6, we demonstrate how branchless narrow systems can be constructed from certain strengthenings of classical square principles. In Section 7, we demonstrate how to force some of these square principles and, in Section 8, we demonstrate how to separate certain of them from one another. In Section 9, we discuss derived systems and use them to get finer control over the failure of the narrow system property. In Section 10, we show that the Proper Forcing Axiom implies that every narrow system with countable width has a cofinal branch but has no effect on narrow systems with uncountable width. At the end, we present some open questions.

Our notation is for the most part standard. The standard reference for undefined notions and notations is [5]. If A is a set of ordinals, then $\text{otp}(A)$ denotes the order type of A and A' denotes the set of limit points of A , i.e. $\{\alpha \in A \mid \sup(A \cap \alpha) = \alpha\}$. If X is a set and κ is a cardinal, then $[X]^\kappa = \{Y \subseteq X \mid |Y| = \kappa\}$. If $\kappa < \lambda$ are regular cardinals, then $S_\kappa^\lambda = \{\alpha < \lambda \mid \text{cf}(\alpha) = \kappa\}$, and $\text{cof}(\kappa)$ denotes the class of ordinals of cofinality κ . On denotes the class of all ordinals. If λ is an uncountable, regular cardinal, T is a λ -tree, and $\alpha < \lambda$, then we will assume that level α of T is $\{\alpha\} \times \kappa_\alpha$, where $\kappa_\alpha < \lambda$. In particular, if $\lambda = \mu^+$, then, for all $0 < \alpha < \lambda$, we will assume that level α of T is $\{\alpha\} \times \mu$. The tree relation for a tree T will typically be denoted by $<_T$. If R is a binary relation, then we will typically write $a <_R b$ to stand for $(a, b) \in R$.

2. NARROW SYSTEMS

Definition Let R be a binary relation on a set X .

- If $a, b \in X$, then a and b are *R-comparable* if $a = b$, $a <_R b$, or $b <_R a$. Otherwise, a and b are *R-incomparable*, which is denoted $a \perp_R b$.
- R is *tree-like* if, for all $a, b, c \in X$, if $a <_R c$ and $b <_R c$, then a and b are *R-comparable*.

We now recall the notion of a λ -system, introduced in [9].

Definition Let λ be an infinite, regular cardinal. $S = \langle \{\{\alpha\} \times \kappa_\alpha \mid \alpha \in I\}, \mathcal{R} \rangle$ is a λ -system if:

- (1) $I \subseteq \lambda$ is unbounded and, for all $\alpha \in I$, $0 < \kappa_\alpha < \lambda$. We sometimes identify S with $\{\{\alpha\} \times \kappa_\alpha \mid \alpha \in I\}$. For each $\alpha \in I$, we say that $S_\alpha := \{\alpha\} \times \kappa_\alpha$ is the α^{th} level of S ;
- (2) \mathcal{R} is a set of binary, transitive, tree-like relations on S and $0 < |\mathcal{R}| < \lambda$;

- (3) for all $R \in \mathcal{R}$, $\alpha_0, \alpha_1 \in I$, $\beta_0 < \kappa_{\alpha_0}$, and $\beta_1 < \kappa_{\alpha_1}$, if $(\alpha_0, \beta_0) <_R (\alpha_1, \beta_1)$, then $\alpha_0 < \alpha_1$;
- (4) for all $\alpha_0 < \alpha_1$, both in I , there are $\beta_0 < \kappa_{\alpha_0}$, $\beta_1 < \kappa_{\alpha_1}$, and $R \in \mathcal{R}$ such that $(\alpha_0, \beta_0) <_R (\alpha_1, \beta_1)$.

If $S = \langle \{\{\alpha\} \times \kappa_\alpha \mid \alpha \in I\}, \mathcal{R} \rangle$ is a λ -system, then we define $\text{width}(S) = \max(\sup(\{\kappa_\alpha \mid \alpha \in I\}, |\mathcal{R}|)$ and $\text{height}(S) = \lambda$. S is a *narrow λ -system* if $\text{width}(S)^+ < \lambda$.

S is a *strong λ -system* if it satisfies the following strengthening of (4):

- (4') for all $\alpha_0 < \alpha_1$, both in I , and for every $\beta_1 < \kappa_1$, there are $\beta_0 < \kappa_0$ and $R \in \mathcal{R}$ such that $(\alpha_0, \beta_0) <_R (\alpha_1, \beta_1)$.

If $R \in \mathcal{R}$, a *branch of S through R* is a set $b \subset S$ such that for all $u, v \in b$, u and v are R -comparable. b is a *cofinal branch* if, for unboundedly many $\alpha \in I$, $b \cap S_\alpha \neq \emptyset$.

Remark If λ is a successor cardinal and $S = \langle \{\{\alpha\} \times \kappa_\alpha \mid \alpha \in I\}, \mathcal{R} \rangle$ is a λ -system, or if κ is weakly inaccessible and S is a narrow λ -system, then there is an unbounded $J \subseteq I$ and a $\kappa < \lambda$ such that, for all $\alpha \in J$, $\kappa_\alpha = \kappa$. It will then be sufficient for us to work with subsystems of the form $\langle J \times \kappa, \mathcal{R} \rangle$, so, in the case that λ is a successor cardinal or we are considering only narrow systems, we will assume our systems are of this form.

Definition Let λ be a regular cardinal, and let T be a λ -tree. T *admits a narrow system* if there is an unbounded $I \subseteq \lambda$ and a κ with $\kappa^+ < \lambda$ such that $\langle I \times \kappa, \{<_T\} \rangle$ is a system.

Remark Note that, as T is a tree, verifying that $\langle I \times \kappa, \{<_T\} \rangle$ in the above definition is a system amounts to checking (4) in the definition of systems, namely that, for all $\alpha_0 < \alpha_1$, both in I , there are $\beta_0, \beta_1 < \kappa$ such that $(\alpha_0, \beta_0) <_T (\alpha_1, \beta_1)$.

We will be interested in statements asserting that all narrow systems of a certain shape have a cofinal branch. We first show that, when discussing cofinal branches through narrow systems, it suffices to consider systems having a single relation.

Proposition 2.1. *Let λ be an uncountable, regular cardinal, and suppose $S = \langle I \times \kappa, \mathcal{R} \rangle$ is a narrow λ -system with no cofinal branch. Suppose $\text{width}(S) = \kappa'$. Then there is a narrow λ -system $S' = \langle I \times \kappa', \mathcal{R}' \rangle$ with no cofinal branch such that $|\mathcal{R}'| = 1$.*

Proof. Let $\kappa_0 = |\mathcal{R}|$, and enumerate \mathcal{R} as $\langle R_\xi \mid \xi < \kappa_0 \rangle$. Fix a bijection $\pi : \kappa' \rightarrow \kappa \times \kappa_0$. For $\beta < \kappa'$, denote $\pi(\beta)$ as (β^0, β^1) . Let $\mathcal{R}' = \{R\}$, and define the system S' by letting $(\alpha_0, \beta_0) <_R (\alpha_1, \beta_1)$ iff $\beta_0^1 = \beta_1^1 =: \xi$ and, in S , $(\alpha_0, \beta_0^0) <_{R_\xi} (\alpha_1, \beta_1^0)$. It is easily verified that S' is a narrow λ -system and that a cofinal branch through S' would give rise to a cofinal branch through S . \square

Definition Let $\kappa \leq \lambda$ be infinite cardinals. The (κ, λ) -*narrow system property* (abbreviated $NSP(\kappa, \lambda)$) holds if every narrow system of width $< \kappa$ and height λ has a cofinal branch. The $(\kappa, \geq \lambda)$ -*narrow system property* ($NSP(\kappa, \geq \lambda)$) holds if every narrow system of width $< \kappa$ and height a regular cardinal $\geq \lambda$ has a cofinal branch.

If λ is a regular, uncountable cardinal, then the *narrow system property* holds at λ (abbreviated $NSP(\lambda)$) if every narrow system of height λ has a cofinal branch. Note that this is the same as $NSP(\lambda, \lambda)$.

Proposition 2.2. *If λ is weakly compact, then $NSP(\lambda)$ holds.*

Proof. Suppose λ is weakly compact, $\kappa < \lambda$, and $S = \langle I \times \kappa, \mathcal{R} \rangle$ is a narrow λ -system. We will show that S has a cofinal branch. By Proposition 2.1, we may assume that $\mathcal{R} = \{R\}$. We define a function $f : [I]^2 \rightarrow \kappa \times \kappa$ as follows. For every $\alpha_0 < \alpha_1$ with $\alpha_0, \alpha_1 \in I$, find $\beta, \gamma \in \kappa$ such that $(\alpha_0, \beta) <_R (\alpha_1, \gamma)$ and let $f(\{\alpha_0, \alpha_1\}) = (\beta, \gamma)$. Since λ is weakly compact, $\lambda \rightarrow (\lambda)_\kappa^2$, so there are an unbounded $H \subseteq I$ and $\beta^*, \gamma^* < \kappa$ such that, for all $\alpha_0 < \alpha_1$ with $\alpha_0, \alpha_1 \in H$, $f(\{\alpha_0, \alpha_1\}) = (\beta^*, \gamma^*)$.

Now let $\alpha_0 < \alpha_1 < \alpha_2$, with all three in H . $(\alpha_0, \beta^*) <_R (\alpha_2, \gamma^*)$ and $(\alpha_1, \beta^*) <_R (\alpha_2, \gamma^*)$, so, since R is tree-like, $(\alpha_0, \beta^*) <_R (\alpha_1, \beta^*)$. Thus, $\{(\alpha, \beta^*) \mid \alpha \in H\}$ is a cofinal branch through S . \square

Proposition 2.3. *If λ is strongly compact, then $NSP(\lambda, \geq \lambda)$ holds.*

Proof. Suppose λ is strongly compact, $\kappa < \lambda$, $\mu \geq \lambda$ is a regular cardinal, and $S = \langle I \times \kappa, \mathcal{R} \rangle$ is a narrow μ -system. We may again assume that $\mathcal{R} = \{R\}$. Since λ is strongly compact, every λ -complete filter can be extended to a λ -complete ultrafilter. Thus, let U be a λ -complete ultrafilter over μ containing I and all co-bounded subsets of μ . As in the proof of Proposition 2.2, define $f : [I]^2 \rightarrow \kappa \times \kappa$ so that, if $\alpha_0 < \alpha_1$ are in I and $f(\{\alpha_0, \alpha_1\}) = (\beta, \gamma)$, then $(\alpha_0, \beta) <_R (\alpha_1, \gamma)$.

For each $\alpha \in I$, use the λ -completeness of U to find $\beta_\alpha, \gamma_\alpha < \kappa$ such that $A_\alpha := \{\alpha' \in I \setminus (\alpha + 1) \mid f(\{\alpha, \alpha'\}) = (\beta_\alpha, \gamma_\alpha)\} \in U$. Again using the λ -completeness of U , find $B \subseteq I$ and $\beta^*, \gamma^* < \kappa$ such that $B \in U$ and, for all $\alpha \in B$, $(\beta_\alpha, \gamma_\alpha) = (\beta^*, \gamma^*)$. Let $\alpha_0 < \alpha_1$ be in B , and let $\alpha_2 \in A_{\alpha_0} \cap A_{\alpha_1}$. Then $(\alpha_0, \beta^*) <_R (\alpha_2, \gamma^*)$ and $(\alpha_1, \beta^*) <_R (\alpha_2, \gamma^*)$, so, since R is tree-like, $(\alpha_0, \beta^*) <_R (\alpha_1, \beta^*)$. Thus, $\{(\alpha, \beta^*) \mid \alpha \in B\}$ is a cofinal branch through S . \square

3. COMBINATORIAL AND FORCING PRELIMINARIES

In this section, we recall some relevant combinatorial and forcing notions and basic facts thereon. We start with variations on Jensen's square principle, which will be important throughout this paper.

Definition Let λ and μ be cardinals, with μ infinite and $\lambda > 1$. A $\square_{\mu, < \lambda}$ -sequence is a sequence $\vec{\mathcal{C}} = \langle \mathcal{C}_\alpha \mid \alpha < \mu^+ \rangle$ such that:

- (1) for all limit $\alpha < \mu^+$, if $C \in \mathcal{C}_\alpha$, then C is a club in α and $\text{otp}(C) \leq \mu$;
- (2) for all limit $\alpha < \mu^+$, $1 \leq |\mathcal{C}_\alpha| < \lambda$;
- (3) for all limit $\alpha < \beta < \mu^+$ and all $C \in \mathcal{C}_\beta$, if $\alpha \in C$, then $C \cap \alpha \in \mathcal{C}_\alpha$.

$\square_{\mu, < \lambda}$ holds if there is a $\square_{\mu, < \lambda}$ -sequence.

Remark $\square_{\mu, < \lambda^+}$ is usually denoted $\square_{\mu, \lambda}$. It is immediate that, if $\lambda_0 < \lambda_1$, then \square_{μ, λ_0} implies \square_{μ, λ_1} . $\square_{\mu, 1}$ is Jensen's classical principle \square_μ . $\square_{\mu, \mu}$ is also called *weak square* and denoted \square_μ^* . \square_μ^* is equivalent to the existence of a special μ^+ -Aronszajn tree. \square_{μ, μ^+} is also called *silly square* and holds in all models of ZFC.

Definition Let $1 < \lambda \leq \kappa$ be cardinals, with κ infinite and regular. A $\square(\kappa, < \lambda)$ -sequence is a sequence $\vec{\mathcal{C}} = \langle \mathcal{C}_\alpha \mid \alpha < \kappa \rangle$ such that:

- (1) for all limit $\alpha < \kappa$, if $C \in \mathcal{C}_\alpha$, then C is a club in α ;
- (2) for all limit $\alpha < \kappa$, $1 \leq |\mathcal{C}_\alpha| < \lambda$;
- (3) for all limit $\alpha < \beta < \kappa$ and all $C \in \mathcal{C}_\beta$, if $\alpha \in C$, then $C \cap \alpha \in \mathcal{C}_\alpha$;
- (4) there is no club $D \subseteq \kappa$ such that, for all $\alpha \in D$, $D \cap \alpha \in \mathcal{C}_\alpha$.

$\square(\kappa, < \lambda)$ holds if there is a $\square(\kappa, < \lambda)$ -sequence. As above, we denote $\square(\kappa, < \lambda^+)$ by $\square(\kappa, \lambda)$ and $\square(\kappa, 1)$ by $\square(\kappa)$.

We will also need the notion of approachability, which plays an important role in the study of successors of singular cardinals.

Definition Let κ be a regular, uncountable cardinal, and let $\vec{a} = \langle a_\alpha \mid \alpha < \kappa \rangle$ be a sequence of bounded subsets of κ . If $\gamma < \kappa$, then γ is *approachable with respect to \vec{a}* if there is an unbounded $A \subseteq \kappa$ such that $\text{otp}(A) = \text{cf}(\gamma)$ and, for every $\beta < \gamma$, there is $\alpha < \gamma$ such that $A \cap \beta = a_\alpha$.

Definition Let κ be a regular, uncountable cardinal.

- If $A \subseteq \kappa$, then $A \in I[\kappa]$ if there is a club $C \subseteq \kappa$ and a sequence $\vec{a} = \langle a_\alpha \mid \alpha < \kappa \rangle$ of bounded subsets of κ such that, for every $\gamma \in A \cap C$, γ is approachable with respect to \vec{a} .
- If $\kappa = \mu^+$, where μ is a singular cardinal, then AP_μ (the *approachability property at μ*) is the assertion that $\kappa \in I[\kappa]$.

$I[\kappa]$ is called the approachability ideal. A wealth of information on $I[\kappa]$ can be found in [4]. We collect some of the relevant facts here.

Remark Let κ be a regular, uncountable cardinal.

- $I[\kappa]$ is a normal, κ -complete ideal extending the non-stationary ideal.
- Suppose $\kappa^{<\kappa} = \kappa$ and $\vec{a} = \langle a_\alpha \mid \alpha < \kappa \rangle$ is a fixed enumeration of all bounded subsets of κ . Then $A \in I[\kappa]$ iff there is a club $C \subseteq \kappa$ such that, for all $\gamma \in A \cap C$, γ is approachable with respect to \vec{a} .
- If μ is an infinite cardinal, then $\square_\mu^* \Rightarrow AP_\mu$.
- If λ is a supercompact cardinal and μ is a singular cardinal such that $\text{cf}(\mu) < \lambda < \mu$, then $\neg AP_\mu$.

We now move to forcing. We first recall the notion of strategic closure.

Definition Let \mathbb{P} be a partial order and let β be an ordinal.

- (1) The two-player game $G_\beta(\mathbb{P})$ is defined as follows: Players I and II alternately play entries in $\langle p_\alpha \mid \alpha < \beta \rangle$, a decreasing sequence of conditions in \mathbb{P} with $p_0 = \mathbb{1}_\mathbb{P}$. Player I plays at odd stages, and Player II plays at even stages (including all limit stages). If there is an even stage $\alpha < \beta$ at which Player II cannot play, then Player I wins. Otherwise, Player II wins.
- (2) \mathbb{P} is said to be *β -strategically closed* if Player II has a winning strategy for the game $G_\beta(\mathbb{P})$. The notion of *$< \beta$ -strategically closed* is defined in the obvious way.

We now introduce the standard forcing poset to add a square sequence.

Definition Let λ and μ be cardinals, with $1 \leq \lambda \leq \mu$ and μ uncountable. $\mathbb{S}(\mu, \lambda)$ is the forcing poset consisting of all conditions $p = \langle \mathcal{C}_\alpha^p \mid \alpha \leq \gamma^p \rangle$ such that:

- (1) $\gamma^p < \mu^+$;
- (2) for all limit $\alpha \leq \gamma^p$ and all $C \in \mathcal{C}_\alpha^p$, C is a club in α and $\text{otp}(C) \leq \mu$;
- (3) for all limit $\alpha \leq \gamma^p$, $1 \leq |\mathcal{C}_\alpha^p| \leq \lambda$;
- (4) for all limit $\alpha < \beta \leq \gamma^p$ and all $C \in \mathcal{C}_\beta^p$, if $\alpha \in C$, then $C \cap \alpha \in \mathcal{C}_\alpha^p$.

For all $p, q \in \mathbb{S}(\mu, \lambda)$, $q \leq p$ iff q end-extends p , i.e. $\gamma^q \geq \gamma^p$ and, for all $\alpha \leq \gamma^p$, $\mathcal{C}_\alpha^q = \mathcal{C}_\alpha^p$.

$\mathbb{S}(\mu, < \lambda)$ is defined similarly, except that, in item (2), we require $1 \leq |\mathcal{C}_\alpha^p| < \lambda$.

Proofs of the following can be found in [2].

Proposition 3.1. *Let λ and μ be cardinals, with $1 \leq \lambda \leq \mu$ and μ uncountable.*

- (1) $\mathbb{S}(\mu, \lambda)$ (resp. $\mathbb{S}(\mu, < \lambda)$), where $1 < \lambda$ is countably-closed and $(\mu + 1)$ -strategically closed. In particular, forcing with $\mathbb{S}(\mu, \lambda)$ (resp. $\mathbb{S}(\mu, < \lambda)$) does not add any new μ -sequences of ordinals.
- (2) Forcing with $\mathbb{S}(\mu, \lambda)$ (resp. $\mathbb{S}(\mu, < \lambda)$) adds a $\square_{\mu, \lambda}$ -sequence (resp. a $\square_{\mu, < \lambda}$ -sequence).

There is also a natural forcing to add a thread through a square sequence.

Definition Suppose λ and μ are cardinals and $\vec{\mathcal{C}} = \langle \mathcal{C}_\alpha \mid \alpha < \mu^+ \rangle$ is a $\square_{\mu, < \lambda}$ -sequence. Let $\kappa \leq \mu$ be a regular, uncountable cardinal. $\mathbb{T}_\kappa(\vec{\mathcal{C}})$ is a forcing poset whose conditions are all t such that:

- (1) t is a closed, bounded subset of μ^+ and $\text{otp}(t) < \kappa$;
- (2) for all $\alpha \in t'$, $t \cap \alpha \in \mathcal{C}_\alpha$.

If $s, t \in \mathbb{T}_\kappa(\vec{\mathcal{C}})$, then $s \leq t$ iff s end-extends t .

In general, $\mathbb{T}_\kappa(\vec{\mathcal{C}})$ can be a very destructive forcing poset. However, if $\vec{\mathcal{C}}$ was added by forcing with $\mathbb{S}(\mu, < \lambda)$, then it is rather nicely behaved. If G is generic for $\mathbb{S}(\mu, < \lambda)$, then $\vec{\mathcal{C}} = \bigcup G$ is a $\square_{\mu, < \lambda}$ -sequence. Let $\vec{\mathcal{C}}$ be an $\mathbb{S}(\mu, < \lambda)$ -name for $\vec{\mathcal{C}}$. Let $\kappa < \mu$ be a regular, uncountable cardinal. For a poset \mathbb{P} and a cardinal θ , let \mathbb{P}^θ denote the full-support product of θ copies of \mathbb{P} .

Proposition 3.2. *Suppose $0 < \theta < \lambda$. Then $\mathbb{S}(\mu, < \lambda) * (\dot{\mathbb{T}}_\kappa(\vec{\mathcal{C}}))^\theta$ has a κ -closed dense subset.*

Proof. Let \mathbb{U} consist of all $(p, \langle \dot{t}_\eta \mid \eta < \theta \rangle) \in \mathbb{S}(\mu, < \lambda) * (\dot{\mathbb{T}}_\kappa(\vec{\mathcal{C}}))^\theta$ such that:

- (1) there is $\langle t_\eta \mid \eta < \theta \rangle \in V$ such that $p \Vdash \langle \dot{t}_\eta \mid \eta < \theta \rangle = \langle \check{t}_\eta \mid \eta < \theta \rangle$;
- (2) for all $\eta < \theta$, $\gamma^p = \max(t_\eta)$.

We first show that \mathbb{U} is dense. To this end, let $(p_0, \langle \dot{t}_\eta^0 \mid \eta < \theta \rangle) \in \mathbb{S}(\mu, < \lambda) * (\dot{\mathbb{T}}_\kappa(\vec{\mathcal{C}}))^\theta$. Since $\mathbb{S}(\mu, < \lambda)$ is μ^+ -distributive, we can find $p \leq p_0$ and $\langle \dot{t}_\eta^0 \mid \eta < \theta \rangle \in V$ such that $p \Vdash \langle \dot{t}_\eta^0 \mid \eta < \theta \rangle = \langle \check{t}_\eta^0 \mid \eta < \theta \rangle$. By strengthening p if necessary, we may assume that $\gamma^p > \max(\dot{t}_\eta^0)$ for all $\eta < \theta$. For all $\eta < \theta$, let $t_\eta = \dot{t}_\eta^0 \cup \{\gamma^p\}$. Then $(p, \langle \check{t}_\eta \mid \eta < \theta \rangle) \leq (p_0, \langle \dot{t}_\eta^0 \mid \eta < \lambda \rangle)$ and is in \mathbb{U} .

We next show that \mathbb{U} is κ -closed. Let $\langle (p_\xi, \langle \dot{t}_\eta^\xi \mid \eta < \theta \rangle) \mid \xi < \epsilon \rangle$ be a strictly decreasing sequence from \mathbb{U} , where $\epsilon < \kappa$ is a limit ordinal. For each $\xi < \epsilon$ and $\eta < \theta$, let t_η^ξ be such that $p_\xi \Vdash \dot{t}_\eta^\xi = \check{t}_\eta^\xi$. Let $\gamma = \sup(\{\gamma^{p_\xi} \mid \xi < \epsilon\})$. For each $\eta < \theta$, let $t_\eta^* = \bigcup_{\xi < \epsilon} t_\eta^\xi$, and note that t_η^* is a club in γ . We define $p = \langle \mathcal{C}_\alpha^p \mid \alpha \leq \gamma \rangle$ as follows. For $\alpha < \gamma$, let $\mathcal{C}_\alpha^p = \mathcal{C}_\alpha^{p_\xi}$ for some $\xi < \epsilon$ such that $\alpha \leq \gamma^{p_\xi}$. Let $\mathcal{C}_\gamma^p = \{t_\eta^* \mid \eta < \theta\}$. Finally, for all $\eta < \theta$, let $t_\eta = t_\eta^* \cup \{\gamma\}$. It is easily verified that $(p, \langle \check{t}_\eta \mid \eta < \theta \rangle)$ is a lower bound for $\langle (p_\xi, \langle \dot{t}_\eta^\xi \mid \eta < \theta \rangle) \mid \xi < \epsilon \rangle$ in \mathbb{U} . \square

Corollary 3.3. *Let G be $\mathbb{S}(\mu, < \lambda)$ -generic over V , and let $\vec{\mathcal{C}} = \bigcup G$. Let $\kappa < \mu$ be a regular, uncountable cardinal.*

- (1) In $V[G]$, $\mathbb{T} = \mathbb{T}_\kappa(\vec{\mathcal{C}})$ is κ -distributive. Moreover, \mathbb{T}^θ is κ -distributive for all $\theta < \lambda$.
- (2) If H is \mathbb{T} -generic over $V[G]$, then $T = \bigcup H$ is a club in $(\mu^+)^V$ of order type κ such that, for every $\alpha \in T'$, $T \cap \alpha \in \mathcal{C}_\alpha$.

Remark A club T as in the previous corollary is said to be a *thread* through $\vec{\mathcal{C}}$.

We similarly introduce a forcing poset to add a $\square(\kappa, \lambda)$ -sequence.

Definition Suppose $1 < \lambda \leq \kappa$ are cardinals, with κ regular and uncountable. $\mathbb{Q}(\kappa, < \lambda)$ is the forcing poset consisting of conditions $q = \langle \mathcal{C}_\alpha^q \mid \alpha \leq \gamma^q \rangle$ such that:

- (1) $\gamma^q < \kappa$;
- (2) for all limit $\alpha \leq \gamma^q$ and all $C \in \mathcal{C}_\alpha^q$, C is a club in α ;
- (3) for all limit $\alpha \leq \gamma^q$, $1 \leq |\mathcal{C}_\alpha^q| < \lambda$;
- (4) for all limit $\alpha < \beta \leq \gamma^q$ and all $C \in \mathcal{C}_\beta^q$, if $\alpha \in C$, then $C \cap \alpha \in \mathcal{C}_\alpha^q$.

$\mathbb{Q}(\kappa, < \lambda)$ is ordered by end-extension. $\mathbb{Q}(\kappa, < \lambda^+)$ will be denoted by $\mathbb{Q}(\kappa, \lambda)$.

Definition Suppose $1 < \lambda \leq \kappa$ are cardinals, with κ regular and uncountable, and $\vec{C} = \langle \mathcal{C}_\alpha \mid \alpha < \kappa \rangle$ is a $\square(\kappa, < \lambda)$ -sequence. $\mathbb{T}(\vec{C})$ is the forcing poset whose conditions are closed, bounded subsets t of κ such that, for all $\alpha \in t'$, $t \cap \alpha \in \mathcal{C}_\alpha$. $\mathbb{T}(\vec{C})$ is ordered by end-extension.

The following is proved similarly to the corresponding facts about forcings to add and thread $\square_{\mu, < \lambda}$ -sequences.

Proposition 3.4. *Let $1 < \lambda \leq \kappa$, with κ regular and uncountable.*

- (1) $\mathbb{Q}(\kappa, < \lambda)$ is countably closed and κ -strategically closed.
- (2) If G is $\mathbb{Q}(\kappa, < \lambda)$ -generic over V , then $\bigcup G$ is a $\square(\kappa, < \lambda)$ -sequence in $V[G]$.
- (3) If \vec{C} is the canonical $\mathbb{Q}(\kappa, < \lambda)$ -name for the union of the generic filter and $\theta < \lambda$, then $\mathbb{Q}(\kappa, < \lambda) * \dot{\mathbb{T}}(\vec{C})^\theta$ has a dense κ -directed closed subset.

We need two more general facts about forcing. The first is due to Magidor [8] and concerns absorbing forcing posets into Levy collapses.

Fact 3.5. *Let κ be a regular cardinal, and let $\kappa < \lambda < \mu$. Suppose that, in $V^{\text{Coll}(\kappa, < \lambda)}$, \mathbb{P} is a separative, κ -closed partial order and $|\mathbb{P}| < \mu$. Let i be the natural complete embedding of $\text{Coll}(\kappa, < \lambda)$ into $\text{Coll}(\kappa, < \mu)$ (namely, the identity embedding). Then i can be extended to a complete embedding j of $\text{Coll}(\kappa, < \lambda) * \mathbb{P}$ into $\text{Coll}(\kappa, < \mu)$ so that the quotient forcing $\text{Coll}(\kappa, < \mu)/j[\text{Coll}(\kappa, < \lambda) * \mathbb{P}]$ is κ -closed.*

The second fact is due to Shelah and involves the preservation of stationary sets by sufficiently closed forcing.

Fact 3.6. *Let $\kappa < \lambda$ be infinite, regular cardinals. Suppose that S is a stationary subset of S_κ^λ , $S \in I[\lambda]$, and \mathbb{P} is a κ^+ -closed forcing poset. Then S remains stationary in $V^\mathbb{P}$.*

4. PRESERVATION LEMMAS

In this section, we present two preservation lemmas, each a slight improvement of a similar lemma of Sinapova [11]. For convenience, we first introduce the following definition.

Definition Let λ be an uncountable regular cardinal, let $S = \langle I \times \kappa, \mathcal{R} \rangle$ be a narrow λ -system, and let $\theta = \text{width}(S)$. $\vec{b} = \{b_\gamma \mid \gamma < \theta\}$ is a *full set of branches through S* if:

- (1) for all $\gamma < \theta$, b_γ is a branch through S ;
- (2) for all $\alpha \in I$, there is $\gamma < \theta$ such that $b_\gamma \cap S_\alpha \neq \emptyset$.

Remark Note that, since λ is regular and $\theta < \lambda$, condition (2) in the above definition implies that, for some $\gamma < \theta$, b_γ is a cofinal branch.

Lemma 4.1. *Suppose that λ is an uncountable cardinal, $S = \langle I \times \kappa, \mathcal{R} \rangle$ is a narrow λ -system, $\text{width}(S) = \theta$, \mathbb{P} is a θ^+ -closed forcing poset, and G is \mathbb{P} -generic over V . If, in $V[G]$, there is a full set of branches through S , then there is a cofinal branch through S in V .*

Remark This lemma improves Sinapova's from [11] in that it applies to θ^+ -closed forcing posets where the original applies only to θ^{++} -closed posets.

Proof. We work in V , supposing for the sake of contradiction that there is no cofinal branch through S . For $\gamma < \theta$, let \dot{b}_γ be a \mathbb{P} -name, and let $p^* \in \mathbb{P}$ be such that $p^* \Vdash \{\dot{b}_\gamma \mid \gamma < \theta\}$ is a full set of branches through S . Since \mathbb{P} is θ^+ -closed, we may assume that there are a nonempty $A \subseteq \theta$, $\alpha^* < \lambda$, and $r : A \rightarrow \mathcal{R}$ such that:

- for all $\gamma < \theta$, $p^* \Vdash \dot{b}_\gamma$ is a cofinal branch iff $\gamma \in A$;
- for all $\gamma \in \theta \setminus A$, $p^* \Vdash \dot{b}_\gamma \subseteq S_{<\alpha^*}$;
- for all $\gamma \in A$, $p^* \Vdash \dot{b}_\gamma$ is a branch through $r(\gamma)$.

Claim 4.2. *For every $p \leq p^*$ and every $\gamma \in A$, there are $q_0, q_1 \leq p$ and $u_0, u_1 \in S$ such that:*

- (1) for $i < 2$, $q_i \Vdash u_i \in \dot{b}_\gamma$;
- (2) $u_0 \perp_{r(\gamma)} u_1$.

Proof. Suppose not, and let p, γ , and β form a counterexample. Then $b = \{u \in S \mid \text{for some } q \leq p, q \Vdash u \in \dot{b}_\gamma\}$ is a cofinal branch through S in V . \square

Claim 4.3. *For every $p_0, p_1 \leq p^*$ and every $\gamma \in A$, there are $q_0 \leq p_0$, $q_1 \leq p_1$, and $u_0, u_1 \in S$ such that:*

- (1) for $i < 2$, $q_i \Vdash u_i \in \dot{b}_\gamma$;
- (2) $u_0 \perp_{r(\gamma)} u_1$.

Proof. First, apply Claim 4.2 to obtain $q_0^0, q_1^0 \leq p_0$ and $u_0^0, u_1^0 \in S$ such that $q_i^0 \Vdash u_i^0 \in \dot{b}_\gamma$ and $u_0^0 \perp_{r(\gamma)} u_1^0$. Let $\beta^* < \lambda$ be such that $u_0^0, u_1^0 \in S_{<\beta^*}$. Find $q_1 \leq p_1$ and $u_1 \in S_{\geq \beta^*}$ such that $q_1 \Vdash u_1 \in \dot{b}_\gamma$. If (u_0^0, u_1) and (u_1^0, u_1) are both in $r(\gamma)$, then, since $r(\gamma)$ is tree-like, u_0^0 and u_1^0 are $r(\gamma)$ -comparable, which is a contradiction. Thus, there is $i^* < 2$ such that $u_{i^*}^0 \perp_{r(\gamma)} u_1$. Let $q_0 = q_{i^*}^0$ and $u_0 = u_{i^*}^0$. Then q_0, q_1, u_0 , and u_1 are as desired. \square

Claim 4.4. *For every $p \leq p^*$, there are $q_0, q_1 \leq p$ and $\{u_i^\gamma \mid \gamma \in A, i < 2\} \subseteq S$ such that:*

- (1) for every $\gamma \in A$ and $i < 2$, $q_i \Vdash u_i^\gamma \in \dot{b}_\gamma$;
- (2) for every $\gamma \in A$, $u_0^\gamma \perp_{r(\gamma)} u_1^\gamma$.

Proof. We recursively build two decreasing sequences, $\langle q_\gamma^0 \mid \gamma < \theta \rangle$ and $\langle q_\gamma^1 \mid \gamma < \theta \rangle$ from \mathbb{P} together with nodes from S , $\{u_i^\gamma \mid \gamma \in A, i < 2\}$, as follows.

First, let $q_0^0 = q_0^1 = p$. If $\gamma < \theta$ is a limit ordinal and $i < 2$, let q_γ^i be a lower bound for $\langle q_\xi^i \mid \xi < \gamma \rangle$. If q_γ^0, q_γ^1 have been defined and $\gamma \notin A$, let $q_{\gamma+1}^i = q_\gamma^i$ for $i < 2$. Finally, if q_γ^0 and q_γ^1 have been defined and $\gamma \in A$, apply Claim 4.3 to q_γ^0, q_γ^1 , and γ to obtain $q_{\gamma+1}^0 \leq q_\gamma^0$, $q_{\gamma+1}^1 \leq q_\gamma^1$, and $u_0^\gamma, u_1^\gamma \in S$ such that:

- for $i < 2$, $q_{\gamma+1}^i \Vdash u_i^\gamma \in \dot{b}_\gamma$;

- $u_0^\gamma \perp_{r(\gamma)} u_1^\gamma$.

At the end of the construction, for $i < 2$, let q_i be a lower bound for $\langle q_\gamma^i \mid \gamma < \theta \rangle$. Then q_0, q_1 , and $\{u_i^\gamma \mid \gamma \in A, i < 2\}$ are as desired. \square

Now use Claim 4.4 and the closure of \mathbb{P} to recursively build a tree of conditions $\{p_\sigma \mid \sigma \in {}^{<\theta}2\}$ and nodes $\{u_i^{\sigma,\gamma} \mid \sigma \in {}^{<\theta}2, \gamma \in A, i < 2\}$ in S as follows.

Let $p_\emptyset = p^*$. If $\eta < \theta$ is a limit ordinal, $\sigma \in {}^\eta 2$, and $p_{\sigma \upharpoonright \xi}$ has been defined for all $\xi < \eta$, let p_σ be a lower bound for $\langle p_{\sigma \upharpoonright \xi} \mid \xi < \eta \rangle$. If $\sigma \in {}^{<\theta}2$ and p_σ has been defined, apply Claim 4.4 to p_σ to obtain $p_{\sigma \smallfrown \langle 0 \rangle}, p_{\sigma \smallfrown \langle 1 \rangle} \leq p_\sigma$ and nodes $\{u_i^{\sigma,\gamma} \mid \gamma \in A, i < 2\}$ in S such that:

- for every $\gamma \in A$ and $i < 2$, $p_{\sigma \smallfrown \langle i \rangle} \Vdash "u_i^{\sigma,\gamma} \in \dot{b}_\gamma"$;
- for every $\gamma \in A$, $u_0^{\sigma,\gamma} \perp_{r(\gamma)} u_1^{\sigma,\gamma}$.

For each $f \in {}^\theta 2$, let p_f be a lower bound for $\langle p_{f \upharpoonright \eta} \mid \eta < \theta \rangle$. Choose $B \subseteq {}^\theta 2$ with $|B| = \theta^+$, and find $\beta^* \in (I \setminus \alpha^*)$ such that $u_i^{f \upharpoonright \eta, \gamma} \in S_{<\beta^*}$ for all $f \in B$, $\eta < \theta$, $\gamma \in A$, and $i < 2$. This is possible, since $\theta^+ < \lambda$ by the assumption that S is a narrow system.

For each $f \in B$, find $q_f \leq p_f$, $v_f \in S_{\beta^*}$, and $\gamma_f \in A$ such that $q_f \Vdash "v_f \in \dot{b}_{\gamma_f}"$. Since $|B| = \theta^+$, we can find $v \in S_{\beta^*}$, $\gamma \in A$, and $f \neq g$, both in B , such that $v_f = v_g = v$ and $\gamma_f = \gamma_g = \gamma$. Let η^* be the least $\eta < \theta$ such that $f(\eta) \neq g(\eta)$, and let $\sigma = f \upharpoonright \eta = g \upharpoonright \eta$. Assume without loss of generality that $f(\eta) = 0$ and $g(\eta) = 1$. Then $q_f \Vdash "u_0^{\sigma,\gamma}, v \in \dot{b}_\gamma"$, so $u_0^{\sigma,\gamma} <_{r(\gamma)} v$. Similarly, $q_g \Vdash "u_1^{\sigma,\gamma}, v \in \dot{b}_\gamma"$, so $u_1^{\sigma,\gamma} <_{r(\gamma)} v$. Thus, since $r(\gamma)$ is tree-like, $u_0^{\sigma,\gamma}$ and $u_1^{\sigma,\gamma}$ are $r(\gamma)$ -comparable, contradicting $u_0^{\sigma,\gamma} \perp_{r(\gamma)} u_1^{\sigma,\gamma}$. \square

The next variation on Sinapova's theorem is due to Neeman [10].

Lemma 4.5. *Suppose that λ is a regular, uncountable cardinal, $S = \langle I \times \kappa, \mathcal{R} \rangle$ is a narrow λ -system, and $\text{width}(S) = \theta$. Suppose \mathbb{P} is a forcing poset, and let \mathbb{P}^{θ^+} denote the full-support product of θ^+ copies of \mathbb{P} . Suppose moreover that \mathbb{P}^{θ^+} is θ^{++} -distributive, G is \mathbb{P} -generic over V , and, in $V[G]$, there is a full set of branches through S . Then there is a cofinal branch through S in V .*

5. WEAK SQUARE AND THE NARROW SYSTEM PROPERTY

In this section, we show that the narrow system property, unlike the tree property, is compatible with certain weak square properties. We first give a general result about forcing the narrow system property at small cardinals.

Theorem 5.1. *Let $\mu < \lambda$ be infinite cardinals, with μ regular and λ supercompact. Let $\mathbb{P} = \text{Coll}(\mu, < \lambda)$. Then, in $V^\mathbb{P}$, $\text{NSP}(\mu, \geq \lambda)$ holds and moreover is indestructible under λ -directed closed set forcing.*

Proof. Let G be \mathbb{P} -generic over V . Since trivial forcing is λ -directed closed, it suffices to prove that if, in $V[G]$, \mathbb{Q} is a λ -directed closed forcing poset and H is \mathbb{Q} -generic over $V[G]$, then $\text{NSP}(\mu, \geq \lambda)$ holds in $V[G * H]$.

Thus, let \mathbb{Q} be λ -directed closed in $V[G]$, and let H be \mathbb{Q} -generic over $V[G]$. In $V[G * H]$, let $\nu \geq \lambda$ be a regular cardinal, let $\kappa < \mu$, and let $S = \langle I \times \kappa, \mathcal{R} \rangle$ be a narrow ν -system. As usual, we assume that $\mathcal{R} = \{R\}$.

In V , let $\dot{\mathbb{Q}}$ be a \mathbb{P} -name for \mathbb{Q} and fix a cardinal $\delta > \nu, |\dot{\mathbb{Q}}|$. Let $j : V \rightarrow M$ witness that λ is δ -supercompact. In particular, $\text{crit}(j) = \lambda$, $j(\lambda) > \delta$, and

${}^\delta M \subseteq M$. $j(\mathbb{P}) = \text{Coll}(\mu, < j(\lambda))$, so, by Fact 3.5, $j(\mathbb{P}) \cong \mathbb{P} * \dot{\mathbb{Q}} * \dot{\mathbb{R}}$, where \mathbb{R} is μ -closed in $V^{\mathbb{P} * \dot{\mathbb{Q}}}$. Thus, letting K be \mathbb{R} -generic over $V[G * H]$, we can extend j to $j : V[G] \rightarrow M[G * H * K]$.

In $M[G * H * K]$, $j(\mathbb{Q})$ is a $j(\lambda)$ -directed closed forcing notion. Moreover, since $M[G * H * K]$ is closed under δ -sequences in $V[G * H * K]$, $j \restriction \mathbb{Q} \in M[G * H * K]$, so $\bar{H} := \{j(q) \mid q \in H\} \in M[G * H * K]$. \bar{H} is a directed subset of $j(\mathbb{Q})$ and $|\bar{H}| < \delta$, so there is $q^* \in j(\mathbb{Q})$ such that $q^* \leq j(q)$ for all $q \in H$. Letting H^+ be $j(\mathbb{Q})$ -generic over $V[G * H * K]$ with $q^* \in H^+$, we may further extend j to $j : V[G * H] \rightarrow M[G * H * K * H^+]$.

In $M[G * H * K * H^+]$, $j(S) = \langle j(I) \times \kappa, \{j(R)\} \rangle$ is a narrow $j(\nu)$ -system. Let $\eta = \sup(j^{\nu}) < \nu$, and let $\xi = \min(j(I) \setminus \eta)$. For each $\gamma < \kappa$, let $I_\gamma = \{\alpha \in I \mid \text{for some } \beta < \kappa, (j(\alpha), \beta) <_{j(R)} (\xi, \gamma)\}$ and, for each $\gamma < \kappa$ and $\alpha \in I_\gamma$, let β_γ^α be the unique β such that $(j(\alpha), \beta) <_{j(R)} (\xi, \gamma)$ and $b_\gamma = \{(\alpha, \beta_\gamma^\alpha) \mid \alpha \in I_\gamma\}$.

$\{b_\gamma \mid \gamma < \kappa\} \in V[G * H * K * H^+]$ and is easily verified to be a full set of branches through S . Note that $j(\mathbb{Q})$ is $j(\lambda)$ -directed closed in $M[G * H * K]$ so, since $M[G * H * K]$ is closed under δ -sequences in $V[G * H * K]$, $j(\mathbb{Q})$ is δ -directed closed in $V[G * H * K]$. Thus, since $\text{width}(S) < \mu$ and, in $V[G * H]$, $\mathbb{R} * j(\dot{\mathbb{Q}})$ is μ -closed, Lemma 4.1 implies that S has a cofinal branch in $V[G * H]$. \square

We can use this to get a universal result.

Theorem 5.2. *Suppose there is a proper class of supercompact cardinals. Then there is a class forcing extension in which every narrow system has a cofinal branch.*

Proof. Let $\langle \kappa_i \mid i \in \text{On} \rangle$ be an increasing, continuous sequence of cardinals such that:

- $\kappa_0 = \omega$;
- if $i = 0$ or i is a successor ordinal, then κ_{i+1} is supercompact;
- if i is a limit ordinal, then $\kappa_{i+1} = \kappa_i^+$.

We may assume that, if i is a limit ordinal, then κ_i is singular, for, if this is not the case, then we may let i be least such that κ_i is regular and work in V_{κ_i} instead of V .

Informally, we force with a class-length iteration of Levy collapses to turn each κ_i into \aleph_i . More precisely, we recursively define posets $\langle \mathbb{P}_i \mid i \in \text{On} \rangle$ as follows.

- \mathbb{P}_0 is trivial forcing.
- If $i = 0$ or i is a successor ordinal, then $\mathbb{P}_{i+1} = \mathbb{P}_i * \text{Coll}(\kappa_i, < \kappa_{i+1})$.
- If i is a limit ordinal, then \mathbb{P}_i is the inverse (i.e. full-support) limit of $\langle \mathbb{P}_j \mid j < i \rangle$ and $\mathbb{P}_{i+1} = \mathbb{P}_i * \{1\}$, where $\{1\}$ is trivial forcing.

For ordinals $i < k$, let $\dot{\mathbb{P}}_{i,k}$ be a \mathbb{P}_i -name such that $\mathbb{P}_k \cong \mathbb{P}_i * \dot{\mathbb{P}}_{i,k}$ and note that, in $V^{\mathbb{P}_i}$, $\mathbb{P}_{i,k}$ is κ_i -directed closed. Thus, $(H(\kappa_i))^{V^{\mathbb{P}_i}} = (H(\kappa_i))^{V^{\mathbb{P}_k}}$, so $V^{\mathbb{P}} = \bigcup_{i \in \text{On}} V^{\mathbb{P}_i}$ is a model of ZFC. Also, standard arguments show that, in $V^{\mathbb{P}}$, for all $i \in \text{On}$, $\kappa_i = \aleph_i$, i.e. $\{\kappa_i \mid i \in \text{On}\}$ are precisely the infinite cardinals of $V^{\mathbb{P}}$. Moreover, for all $i \in \text{On}$, κ_i is regular in $V^{\mathbb{P}}$ iff i is 0 or a successor ordinal.

We now show that, in $V^{\mathbb{P}}$, every narrow system has a cofinal branch. It suffices to show that, for all ordinals i, k with $i + 1 < k$ and k a successor ordinal, every narrow system with width κ_i and height κ_k has a cofinal branch. Since, in $V^{\mathbb{P}_{k+1}}$, $\mathbb{P}_{k+1, \ell}$ is κ_{k+1} -closed for all $\ell \geq k + 1$, all such narrow systems are in $V^{\mathbb{P}_{k+1}}$. We may as usual just deal with narrow systems with a single relation.

In $V^{\mathbb{P}_{i+1}}$, $\kappa_{i+1} = \kappa_i^+$ and κ_{i+2} is supercompact. $\mathbb{P}_{i+1, i+2} = \text{Coll}(\kappa_{i+1}, < \kappa_{i+2})$, so, by Theorem 5.1, $NSP(\kappa_{i+1}, \geq \kappa_{i+2})$ holds in $V^{\mathbb{P}_{i+2}}$ and is indestructible under κ_{i+2} -directed closed set forcing. Since $\mathbb{P}_{i+2, k+1}$ is κ_{i+2} -directed closed, $NSP(\kappa_{i+1}, \geq \kappa_{i+2})$ holds in $V^{\mathbb{P}_{k+1}}$. Since $\kappa_k \geq \kappa_{i+2}$, we have that, in $V^{\mathbb{P}_{k+1}}$, every narrow system with width κ_i and height κ_k has a cofinal branch. \square

Theorem 5.3. *Suppose there are infinitely many supercompact cardinals. Then there is a forcing extension in which $\square_{\aleph_\omega, < \aleph_\omega}$ and the narrow system property at $\aleph_{\omega+1}$ both hold.*

Proof. Let $\langle \kappa_n \mid n < \omega \rangle$ be an increasing sequence of supercompact cardinals, let $\mu = \sup(\{\kappa_n \mid n < \omega\})$, and let $\lambda = \mu^+$. Define an iteration $\langle \mathbb{P}_n, \dot{\mathbb{Q}}_n \mid n < \omega \rangle$ by letting $\mathbb{Q}_0 = \text{Coll}(\omega, < \kappa_0)$ and, for all $n < \omega$, letting $\dot{\mathbb{Q}}_{n+1}$ be a \mathbb{P}_{n+1} -name for $\text{Coll}(\kappa_n, < \kappa_{n+1})$. Let \mathbb{P} be the inverse limit of the iteration. Thus, in $V^{\mathbb{P}}$, $\kappa_n = \aleph_{n+1}$ for all $n < \omega$, $\mu = \aleph_\omega$, and $\lambda = \aleph_{\omega+1}$. For all $n < \omega$, let $\dot{\mathbb{P}}^n$ be a \mathbb{P}_n -name such that $\mathbb{P} \cong \mathbb{P}_n * \dot{\mathbb{P}}^n$ and, for all $m < n < \omega$, let $\dot{\mathbb{P}}_{mn}$ be a \mathbb{P}_m name such that $\mathbb{P}_n \cong \mathbb{P}_m * \dot{\mathbb{P}}_{mn}$.

In $V^{\mathbb{P}}$, let $\mathbb{S} = \mathbb{S}(\mu, < \mu)$, the forcing to add a $\square_{\mu, < \mu}$ -sequence. We claim that $V^{\mathbb{P} * \dot{\mathbb{S}}}$ is the desired model. To this end, let G be \mathbb{P} -generic over V , and let H be \mathbb{S} -generic over $V[G]$. For $n < \omega$, let G_n and G^n be the generic filters induced by G on \mathbb{P}_n and \mathbb{P}^n , respectively. For $m < n < \omega$, let G_{mn} be the generic filter induced by G on \mathbb{P}_{mn} . Suppose for sake of contradiction that, in $V[G * H]$, there is a narrow λ -system with no cofinal branch. By proposition 2.1, we may assume that there is such a system of the form $\langle \lambda \times \kappa, \{R\} \rangle$ for some $\kappa < \lambda$.

Let $\vec{C} = \bigcup H$. $\vec{C} = \langle C_\alpha \mid \alpha < \lambda \rangle$ is thus a $\square_{\mu, < \mu}$ -sequence in $V[G * H]$. Let $n^* < \omega$ be least such that $\kappa \leq \kappa_{n^*}$, let $\kappa^* = \kappa_{n^*+3}$, and let $\mathbb{T} = \mathbb{T}_{\kappa^*}(\vec{C})$. Since $|\mathbb{P}_{n^*+3}| < \kappa^*$, κ^* remains supercompact in $V[G_{n^*+3}]$. Let $j : V[G_{n^*+3}] \rightarrow M[G_{n^*+3}]$ witness that κ^* is λ -supercompact, i.e. $\text{crit}(j) = \kappa^*$, $j(\kappa^*) > \lambda$, and ${}^\lambda M[G_{n^*+3}] \subseteq M[G_{n^*+3}]$. Note that $j(\mathbb{Q}_{n^*+3}) = j(\text{Coll}(\kappa_{n^*+2}, < \kappa^*)) = \text{Coll}(\kappa_{n^*+2}, < j(\kappa^*))$. In $V[G_{n^*+4}]$, $\mathbb{P}^{n^*+4} * \dot{\mathbb{S}} * \mathbb{T}$ has a dense κ_{n^*+2} -closed subset and is of size less than $j(\kappa^*)$, so, by Fact 3.5, in $V[G_{n^*+3}]$, $\text{Coll}(\kappa_{n^*+2}, < j(\kappa^*)) \cong \mathbb{P}^{n^*+3} * \dot{\mathbb{S}} * \dot{\mathbb{T}} * \dot{\mathbb{R}}$, where $\dot{\mathbb{R}}$ is forced to be κ_{n^*+2} -closed. Thus, letting I be \mathbb{T} -generic over $V[G * H]$ and letting J be \mathbb{R} -generic over $V[G * H * I]$, we may extend j to a map $j : V[G_{n^*+4}] \rightarrow M[G * H * I * J]$.

We would like to extend j further to have domain $V[G * H]$. To do this, we will construct a master condition in $j(\mathbb{P}^{n^*+4} * \mathbb{S})$. Working in $M[G * H * I * J]$, we first define a condition $p^* \in j(\mathbb{P}^{n^*+4})$. We will think of p^* as a function with domain $[n^* + 4, \omega)$. For each $i \in [n^* + 4, \omega)$, let $p^*(i) = \bigcup_{p \in G} j(p(i))$. Since $|G| = \lambda$, $M[G * H * I * J]$ is closed under λ -sequences, and each iterand of $j(\mathbb{P}^{n^*+4})$ is forced to be $j(\kappa^*)$ -directed closed, we have $p^* \in M[G * H * I * J] \cap j(\mathbb{P}^{n^*+4})$. Moreover, $p^* \leq j(p)$ for all $p \in G^{n^*+4}$. Thus, letting G^+ be $j(\mathbb{P}^{n^*+4})$ -generic over $V[G * H * I * J]$ with $p^* \in G^+$, we can lift j to $j : V[G] \rightarrow M[G * H * I * J * G^+]$. Let $\eta = \sup(j \restriction \lambda)$. In $M[G * H * I * J * G^+]$, let $q^* = \bigcup_{q \in H} j(q)$. q^* is then of the form $\langle C_\alpha^{q^*} \mid \alpha < \eta \rangle$ and is almost an element of $j(\mathbb{S})$; all it lacks is a top element. As with p^* , $q^* \in M[G * H * I * J * G^+]$ by closure under λ -sequences. Let $D = \bigcup I$, and let $E = j \restriction D$. Since D is club in λ of order-type κ^* and j is continuous at points of cofinality less than κ^* , we have that E is a club in η . Moreover, since D is a thread through \mathcal{C} , we have $E \cap \alpha \in C_\alpha^{q^*}$ for all $\alpha \in E'$. Thus, if we define $q^{**} = \langle C_\alpha^{q^{**}} \mid \alpha \leq \eta \rangle$ by letting $C_\alpha^{q^{**}} = C_\alpha^{q^*}$ for all $\alpha < \eta$ and $C_\eta^{q^{**}} = \{E\}$, we

have $q^{**} \in M[G * H * I * J * G^+] \cap j(\mathbb{S})$ and $q^{**} \leq j(q)$ for all $q \in H$. Letting H^+ be $j(\mathbb{S})$ -generic over $V[G * H * I * J * G^+]$ with $q^{**} \in H^+$, we can extend j to $j : V[G * H] \rightarrow M[G * H * I * J * G^+ * H^+]$.

In $M[G * H * I * J * G^+ * H^+]$, $j(S) = \langle j(\lambda) \times \kappa, \{j(R)\} \rangle$ is a narrow $j(\lambda)$ -system. Recall that $\eta = \sup(j \restriction \lambda)$. For each $\gamma < \kappa$, let $b_\gamma = \{u \in S \mid j(u) <_{j(R)} (\eta, \gamma)\}$. It is easily verified that $\{b_\gamma \mid \gamma < \kappa\}$ is a full set of branches through S .

$\{b_\gamma \mid \gamma < \kappa\} \in V[G * H * I * J * G^+ * H^+]$ and, since $G^+ * H^+$ is generic for $j(\kappa^*)$ -distributive forcing, we actually have $\{b_\gamma \mid \gamma < \kappa\} \in V[G * H * I * J]$. Moreover, by Proposition 3.2 and the fact that, in $V[G * H * I]$, \mathbb{R} is κ_{n^*+2} -closed, we have that, in $V[G * H]$, $(\mathbb{T} * \mathbb{R})^{\kappa_{n^*+1}}$ is κ_{n^*+2} -distributive. Therefore, by Lemma 4.5, there is a cofinal branch through S in $V[G * H]$. \square

Remark Since $\square_{\aleph_\omega, < \aleph_\omega}$ implies the existence of a special $\aleph_{\omega+1}$ -Aronszajn tree, it must be the case that in the model for the previous theorem, there are $\aleph_{\omega+1}$ -trees that do not admit narrow systems.

We can use a similar argument to show that, unlike the tree property, the narrow system property is not equivalent to weak compactness for inaccessible cardinals.

Theorem 5.4. *Suppose λ is a supercompact cardinal. There is a forcing extension in which λ remains inaccessible, $NSP(\lambda, \geq \lambda)$ holds, and λ is not weakly compact.*

Proof. By forcing with the Laver preparation (see [7]), we may assume that the supercompactness of λ is indestructible under λ -directed closed forcing. Let $\mathbb{Q} = \mathbb{Q}(\lambda, < \lambda)$, the standard forcing to add a $\square(\lambda, < \lambda)$ -sequence introduced in Section 3. Let G be \mathbb{Q} -generic over V . In $V[G]$, let $\vec{C} = \bigcup G$, and let $\mathbb{T} = \mathbb{T}(\vec{C})$ be the standard forcing to add a thread through \vec{C} of order-type λ . Recall that, in V , $\mathbb{Q} * \dot{\mathbb{T}}^\theta$ has a dense λ -directed closed subset for all $0 < \theta < \lambda$. Let H be \mathbb{T} -generic over $V[G]$.

Our desired model is $V[G]$. Note that λ remains inaccessible in $V[G]$ and, since $\square(\lambda, < \lambda)$ holds, λ is not weakly compact. Suppose for sake of contradiction that $S = \langle I \times \kappa, \{R\} \rangle$ is a narrow ν -system in $V[G]$ such that $\kappa < \lambda \leq \nu$, ν is regular, and S has no cofinal branch. In $V[G * H]$, λ is supercompact. Fix $j : V[G * H] \rightarrow M$ witnessing that λ is ν -supercompact. As in the proof of Theorem 5.3, use j to argue that S has a full set of branches in $V[G * H]$ and then, again as in the proof of Theorem 5.3, argue that this implies that S has a cofinal branch in $V[G]$. \square

6. COUNTEREXAMPLES TO THE NARROW SYSTEM PROPERTY

In this section, we construct narrow λ -systems with no cofinal branches from certain subadditive functions with domain $[\lambda]^2$. We then show that the existence of such functions follows from modified square principles.

Proposition 6.1. *Suppose κ and λ are cardinals, with $\kappa^+ < \lambda$ and λ regular, and suppose there is $d : [\lambda]^2 \rightarrow \kappa$ satisfying:*

- (1) *for all $\alpha < \beta < \gamma < \lambda$, $d(\alpha, \gamma) \leq \max(d(\alpha, \beta), d(\beta, \gamma))$;*
- (2) *for all $\alpha < \beta < \gamma < \lambda$, $d(\alpha, \beta) \leq \max(d(\alpha, \gamma), d(\beta, \gamma))$;*
- (3) *for all unbounded $I \subseteq \lambda$, $d \restriction [I]^2$ is unbounded in κ .*

Then there is a narrow λ -system $S = \langle \lambda \times \kappa, \{R\} \rangle$ with no cofinal branch.

Proof. To define the narrow λ -system $S = \langle \lambda \times \kappa, \{R\} \rangle$, we only need to specify the relation R . Given $\alpha_0 < \alpha_1 < \lambda$ and $\beta_0, \beta_1 < \kappa$, let $(\alpha_0, \beta_0) <_R (\alpha_1, \beta_1)$ iff

$\beta_0 = \beta_1 \geq d(\{\alpha_0, \alpha_1\})$. It is simple to check that S as defined is a narrow λ -system; the fact that R is transitive follows from property (1) of d , and the fact that R is tree-like follows from property (2) of d . The fact that S has no cofinal branch follows from property (3) of d . \square

Remark A function $d : [\lambda]^2 \rightarrow \kappa$ satisfying (1) and (2) from the statement of Proposition 6.1 is called *subadditive*. A function satisfying (3) is called *unbounded*.

We now introduce two different modifications of $\square(\lambda)$ and show that each implies the existence of such subadditive functions. The first is a variant of *indexed square*, a notion studied in [3] and [2].

Definition Let $\kappa < \lambda$ be infinite regular cardinals. A $\square^{\text{ind}}(\lambda, \kappa)$ -sequence is a matrix $\vec{C} = \langle C_{\alpha, i} \mid \alpha < \lambda, i(\alpha) \leq i < \kappa \rangle$ satisfying the following conditions.

- (1) For all limit $\alpha < \lambda$, $i(\alpha) < \kappa$.
- (2) For all limit $\alpha < \lambda$ and $i(\alpha) \leq i < \kappa$, $C_{\alpha, i}$ is a club in α .
- (3) For all limit $\alpha < \lambda$ and $i(\alpha) \leq i < j < \kappa$, $C_{\alpha, i} \subseteq C_{\alpha, j}$.
- (4) For all limit $\alpha < \beta < \lambda$ and $i(\beta) \leq i < \kappa$, if $\alpha \in C'_{\beta, i}$, then $i(\alpha) \leq i$ and $C_{\beta, i} \cap \alpha = C_{\alpha, i}$.
- (5) For all limit $\alpha < \beta < \lambda$, there is $i < \kappa$ such that $\alpha \in C'_{\beta, i}$ (and hence $\alpha \in C'_{\beta, j}$ for all $i \leq j < \kappa$).
- (6) There is no club $D \subseteq \lambda$ such that, for all $\alpha \in D'$, there is $i < \kappa$ such that $D' \cap \alpha = C_{\alpha, i}$. (Such a club D would be called a *thread through \vec{C}* .)

$\square^{\text{ind}}(\lambda, \kappa)$ is the assertion that there is a $\square^{\text{ind}}(\lambda, \kappa)$ -sequence.

Proposition 6.2. *The above definition is unchanged if we replace condition (6) by the following seemingly weaker condition:*

There is no club $D \subseteq \lambda$ and $i < \kappa$ such that, for all $\alpha \in D'$, $D \cap \alpha = C_{\alpha, i}$.

Proof. Suppose $\vec{C} = \langle C_{\alpha, i} \mid \alpha < \lambda, i(\alpha) \leq i < \kappa \rangle$ satisfies conditions (1)-(5) as above and there is a club $D \subseteq \lambda$ such that, for every $\alpha \in D'$, there is $i < \kappa$ such that $D \cap \alpha = C_{\alpha, i}$. For each $\alpha \in D'$, let $i(\alpha)$ be such an i . Since $\kappa < \lambda$ and λ is regular, there is an unbounded $A \subseteq D'$ and an $i^* < \kappa$ such that, for all $\alpha \in A$, $i(\alpha) = i^*$. We claim that, for all $\alpha \in D'$, $D \cap \alpha = C_{\alpha, i^*}$. To see this, fix $\alpha \in D'$, and find $\beta \in A \setminus \alpha$. Then $D \cap \beta = C_{\beta, i^*}$, so $\alpha \in C'_{\beta, i^*}$. Thus, by condition (4), $D \cap \alpha = C_{\beta, i^*} \cap \alpha = C_{\alpha, i^*}$. \square

We will deal with the consistency of $\square^{\text{ind}}(\lambda, \kappa)$ in a later section. We show now that it easily gives rise to subadditive, unbounded functions.

Proposition 6.3. *Suppose $\kappa < \lambda$ are infinite regular cardinals and $\square^{\text{ind}}(\lambda, \kappa)$ holds. Then there is a subadditive, unbounded function $d : [\lambda]^2 \rightarrow \kappa$.*

Proof. Let π be the unique order-preserving bijection from λ to $\lim(\lambda)$. For all $\alpha < \beta < \lambda$, let $d(\alpha, \beta)$ be the least $i < \kappa$ such that $\pi(\alpha) \in C'_{\pi(\beta), i}$. We now verify that d satisfies conditions (1)-(3) from the statement of Proposition 6.1.

To see (1), let $\alpha < \beta < \gamma < \lambda$, and suppose $i \geq \max(d(\alpha, \beta), d(\beta, \gamma))$. Since $i \geq d(\alpha, \beta)$, $\pi(\alpha) \in C'_{\pi(\beta), i}$ and, since $i \geq d(\beta, \gamma)$, $C_{\pi(\gamma), i} \cap \pi(\beta) = C_{\pi(\beta), i}$. Thus, $\pi(\alpha) \in C'_{\pi(\gamma), i}$, so $i \geq d(\alpha, \gamma)$.

To see (2), let $\alpha < \beta < \gamma < \lambda$, and suppose $i \geq \max(d(\alpha, \gamma), d(\beta, \gamma))$. As above, this implies that $\pi(\alpha) \in C'_{\pi(\gamma), i}$ and $C_{\pi(\gamma), i} \cap \pi(\beta) = C_{\pi(\beta), i}$, so $\pi(\alpha) \in C'_{\pi(\beta), i}$ and hence $i \geq d(\alpha, \beta)$.

Finally, to check (3), suppose for sake of contradiction that $I \subseteq \lambda$ is unbounded and $j < \kappa$ is such that $d^{\omega}[I]^2 \subseteq j$. Then, if $\alpha < \beta < \lambda$ and $\alpha, \beta \in I$, we have $\pi(\alpha) \in C'_{\pi(\beta),j}$, so $C_{\pi(\beta),j} \cap \alpha = C_{\pi(\alpha),j}$. Let $D = \bigcup_{\alpha \in A} C_{\pi(\alpha),j}$. D is a club in λ . Let $\alpha < \lambda$ be such that $\pi(\alpha) \in D'$, and let $\beta \in I \setminus \alpha$. Then $D \cap \pi(\beta) = C_{\pi(\beta),j}$, so $\pi(\alpha) \in C'_{\pi(\beta),j}$ and hence $D \cap \pi(\alpha) = C_{\pi(\beta),j} \cap \pi(\alpha) = C_{\pi(\alpha),j}$, so D is a thread through \vec{C} , which is a contradiction. \square

The second square variation we consider is one in which we make slight additional demands on the order types of the clubs.

Definition Let $\kappa < \lambda$ be infinite regular cardinals. A $\square^\kappa(\lambda)$ -sequence is a $\square(\lambda)$ -sequence $\vec{C} = \langle C_\alpha \mid \alpha < \lambda \rangle$ such that, for stationarily many $\alpha \in S_\kappa^\lambda$, $\text{otp}(C_\alpha) < \alpha$. $\square^\kappa(\lambda)$ is the assertion that there is a $\square^\kappa(\lambda)$ -sequence.

The proofs of Propositions 3.3-3.5 in [6] yield the following results.

Proposition 6.4. *Let $\kappa < \lambda$ be infinite regular cardinals. The following are equivalent:*

- (1) $\square^\kappa(\lambda)$ holds;
- (2) *there is a $\square(\lambda)$ -sequence $\vec{C} = \langle C_\alpha \mid \alpha < \lambda \rangle$ and a stationary set $S \subseteq S_\kappa^\lambda$ such that, for all $\alpha \in S$:*
 - $\text{otp}(C_\alpha) = \kappa$;
 - for all $\alpha < \beta < \lambda$, $\alpha \notin C'_\beta$.

Proposition 6.5. *If $\theta < \kappa < \lambda$ are infinite regular cardinals, then $\square^\kappa(\lambda) \Rightarrow \square^\theta(\lambda)$.*

We now introduce a function, due to Todorćević, that can be derived from a $\square(\lambda)$ -sequence. First, define $\Lambda_\kappa : [\lambda]^2 \rightarrow \lambda$ by

$$\Lambda_\kappa(\alpha, \beta) = \max\{\xi \in C_\beta \cap (\alpha + 1) \mid \kappa \text{ divides } \text{otp}(C_\beta \cap \xi)\}$$

Next, let $\rho_\kappa : [\lambda]^2 \rightarrow \kappa$ be defined recursively by

$$\begin{aligned} \rho_\kappa(\alpha, \beta) &= \sup\{\text{otp}(C_\beta \cap [\Lambda_\kappa(\alpha, \beta), \alpha)), \rho_\kappa(\alpha, \min(C_\beta \setminus \alpha)), \\ &\quad \rho_\kappa(\xi, \alpha) \mid \xi \in C_\beta \cap [\Lambda_\kappa(\alpha, \beta), \alpha)\} \end{aligned}$$

The proof of the following proposition can be found in [12].

Proposition 6.6. *Let $\kappa < \lambda$ be infinite regular cardinals, let \vec{C} be a $\square(\lambda)$ -sequence, and let ρ_κ be derived as above from \vec{C} .*

- (1) *For all $\alpha < \beta < \gamma < \lambda$, $\rho_\kappa(\alpha, \gamma) \leq \max(\rho_\kappa(\alpha, \beta), \rho_\kappa(\beta, \gamma))$.*
- (2) *For all $\alpha < \beta < \gamma < \lambda$, $\rho_\kappa(\alpha, \beta) \leq \max(\rho_\kappa(\alpha, \gamma), \rho_\kappa(\beta, \gamma))$.*
- (3) *If \vec{C} is as in (2) of Proposition 6.4, then, for all unbounded $I \subseteq \lambda$, $\rho_\kappa^{\omega}[I]^2$ is unbounded in κ .*

7. FORCING INDEXED SQUARE

In this section, we demonstrate how to force the existence of a $\square^{\text{ind}}(\lambda, \kappa)$ -sequence.

Definition Let $\kappa < \lambda$ be infinite regular cardinals. $\mathbb{P}(\lambda, \kappa)$ is a forcing poset with conditions $p = \langle C_{\alpha,i}^p \mid \alpha \leq \gamma^p, i(\alpha)^p \leq i < \kappa \rangle$ satisfying the following conditions.

- (1) $\gamma^p < \lambda$ is a limit ordinal and, for all limit $\alpha \leq \gamma^p$, $i(\alpha)^p < \kappa$.
- (2) For all limit $\alpha \leq \gamma^p$ and $i(\alpha)^p \leq i < \kappa$, $C_{\alpha,i}^p$ is a club in α .
- (3) For all limit $\alpha \leq \gamma^p$ and $i(\alpha)^p \leq i < j < \kappa$, $C_{\alpha,i}^p \subseteq C_{\alpha,j}^p$.

- (4) For all limit $\alpha < \beta \leq \gamma^p$ and $i(\beta)^p \leq i < \kappa$, if $\alpha \in (C_{\beta,i}^p)'$, then $i(\alpha)^p \leq i$ and $C_{\beta,i}^p \cap \alpha = C_{\alpha,i}^p$.
- (5) For all limit $\alpha < \beta \leq \gamma^p$, there is $i < \kappa$ such that $\alpha \in (C_{\beta,i}^p)'$.

If $p, q \in \mathbb{P}(\lambda, \kappa)$, then $q \leq p$ iff q end-extends p , i.e.:

- $\gamma^q \geq \gamma^p$.
- For all limit $\alpha \leq \gamma^p$, $i(\alpha)^q = i(\alpha)^p$.
- For all limit $\alpha \leq \gamma^p$ and $i(\alpha)^p \leq i < \kappa$, $C_{\alpha,i}^q = C_{\alpha,i}^p$.

Proposition 7.1. $\mathbb{P}(\lambda, \kappa)$ is κ -directed closed.

Proof. Let $\mathbb{P} = \mathbb{P}(\lambda, \kappa)$. Since \mathbb{P} is tree-like, i.e. if $r \leq p, q$, then p and q are comparable, it suffices to verify that \mathbb{P} is κ -closed. To this end, let $\xi < \kappa$, and let $\vec{p} = \langle p_\eta \mid \eta < \xi \rangle$ be a decreasing sequence from \mathbb{P} . Without loss of generality, we may assume that \vec{p} is strictly decreasing and ξ is a limit ordinal.

Let $\gamma = \sup(\{\gamma^{p_\eta} \mid \eta < \xi\})$. We will define q , a lower bound for \vec{p} , so that $q = \langle C_{\alpha,i}^q \mid \alpha \leq \gamma, i(\alpha)^q \leq i < \kappa \rangle$. For limit $\alpha < \gamma$, let $\eta < \xi$ be such that $\alpha \leq \gamma^{p_\eta}$, let $i(\alpha)^q = i(\alpha)^{p_\eta}$ and, for $i(\alpha)^q \leq i < \kappa$, let $C_{\alpha,i}^q = C_{\alpha,i}^{p_\eta}$. It remains to define $i(\gamma)^q$ and $C_{\gamma,i}^q$ for $i(\gamma)^q \leq i < \kappa$.

For $\eta < \zeta < \xi$, let $i(\eta, \zeta)$ be the least $i < \kappa$ such that $\gamma^{p_\eta} \in (C_{\gamma^{p_\zeta}, i}^{p_\zeta})'$. Let $i^* = \sup(\{i(\eta, \zeta) \mid \eta < \zeta < \xi\})$. Since $\xi < \kappa$ and κ is regular, $i^* < \kappa$. Also, for all $i^* \leq i < \kappa$ and all $\eta < \zeta < \xi$, $C_{\gamma^{p_\zeta}, i}^{p_\zeta} \cap \gamma^{p_\eta} = C_{\gamma^{p_\eta}, i}^{p_\eta}$. Thus, letting $i(\gamma)^q = i^*$ and, for all $i^* \leq i < \kappa$, $C_{\gamma,i}^q = \bigcup_{\eta < \xi} C_{\gamma^{p_\eta}, i}^{p_\eta}$, it is easily verified that $q \in \mathbb{P}$ and is a lower bound for \vec{p} . \square

Proposition 7.2. $\mathbb{P}(\lambda, \kappa)$ is λ -strategically closed.

Proof. Let $\mathbb{P} = \mathbb{P}(\lambda, \kappa)$. We describe a winning strategy for Π in $G_\lambda(\mathbb{P})$. Suppose $0 < \xi < \lambda$ is an even ordinal and $\langle p_\eta \mid \eta < \xi \rangle$ is a partial play of $G_\lambda(\mathbb{P})$. Assume we have arranged inductively that, for all even $0 < \eta < \xi$, $i(\gamma^{p_\eta})^{p_\eta} = 0$ and, for all even $\eta_0 < \eta$, $\eta_0 \in (C_{\gamma^{p_{\eta_0}}, 0}^{p_{\eta_0}})'$.

Suppose first that $\xi = \eta + 2$ for some even $\eta < \lambda$. Let $\gamma = \gamma^{p_{\eta+1}} + \omega$. We will define $p_\xi \leq p_{\eta+1}$ so that $p_\xi = \langle C_{\alpha,i}^{p_\xi} \mid \alpha \leq \gamma, i(\alpha)^{p_\xi} \leq i < \kappa \rangle$. Since p_ξ must be an end-extension of $p_{\eta+1}$, we need only define $i(\gamma)^{p_\xi}$ and $C_{\gamma,i}^{p_\xi}$ for $i(\gamma)^{p_\xi} \leq i < \kappa$. As required to maintain the inductive hypothesis, we set $i(\gamma)^{p_\xi} = 0$. For $i < \kappa$ such that $\gamma^{p_\eta} \notin (C_{\gamma^{p_{\eta+1}}, i}^{p_{\eta+1}})'$, we let $C_{\gamma,i}^{p_\xi} = C_{\gamma^{p_\eta}, i}^{p_\eta} \cup \{\gamma^{p_\eta} + n \mid n < \omega\}$. For $i < \kappa$ such that $\gamma^{p_\eta} \in (C_{\gamma^{p_{\eta+1}}, i}^{p_{\eta+1}})'$, we let $C_{\gamma,i}^{p_\xi} = C_{\gamma^{p_{\eta+1}}, i}^{p_{\eta+1}} \cup \{\gamma^{p_{\eta+1}} + n \mid n < \omega\}$. It is easily verified that $p_\xi \leq p_{\eta+1}$ and maintains the inductive hypothesis.

Next, suppose that ξ is a limit ordinal. Let $\gamma = \sup(\{\gamma^{p_\eta} \mid \eta < \xi\})$. We define p_ξ to be a lower bound for $\langle p_\eta \mid \eta < \xi \rangle$ of the form $\langle C_{\alpha,i}^{p_\xi} \mid \alpha \leq \gamma, i(\alpha)^{p_\xi} \leq i < \kappa \rangle$. Again, we only have to specify $i(\gamma)^{p_\xi}$ and $C_{\gamma,i}^{p_\xi}$ for $i(\gamma)^{p_\xi} \leq i < \kappa$. Let $i(\gamma)^{p_\xi} = 0$ and, for all $i < \kappa$, let $C_{\gamma,i}^{p_\xi} = \bigcup \{C_{\gamma^{p_\eta}, i}^{p_\eta} \mid \eta < \xi, \eta \text{ even}\}$. It is again easily verified that p_ξ is a lower bound for $\langle p_\eta \mid \eta < \xi \rangle$ and maintains the inductive hypothesis. \square

Corollary 7.3. Forcing with $\mathbb{P}(\lambda, \kappa)$ preserves all cardinalities and cofinalities $\leq \lambda$. If, in addition, $\lambda^{<\lambda} = \lambda$, then $|\mathbb{P}(\lambda, \kappa)| = \lambda$ and hence preserves all cardinalities and cofinalities.

A variation on the proof of Proposition 7.2 yields the following.

Proposition 7.4. *Let $\alpha < \lambda$, and let $D_\alpha = \{p \in \mathbb{P}(\lambda, \kappa) \mid \alpha \leq \gamma^p\}$. Then D_α is dense in $\mathbb{P}(\lambda, \kappa)$.*

Proposition 7.5. *Let G be $\mathbb{P}(\lambda, \kappa)$ -generic over V . Let $\vec{C} = \bigcup G = \langle C_{\alpha, i} \mid \alpha < \lambda, i(\alpha) \leq i < \kappa \rangle$. Then \vec{C} is a $\square^{\text{ind}}(\lambda, \kappa)$ -sequence.*

Proof. Let $\mathbb{P} = \mathbb{P}(\lambda, \kappa)$. The fact that \vec{C} satisfies conditions (1)-(5) in the definition of $\square^{\text{ind}}(\lambda, \kappa)$ follows easily from the definition of \mathbb{P} . Thus, we only need to verify that \vec{C} satisfies condition (6) or, alternatively, the condition identified in Proposition 6.2. To this end, suppose for sake of contradiction that there is a club $D \subseteq \lambda$ and an $i^* < \kappa$ such that, for all $\alpha \in D'$, $D \cap \alpha = C_{\alpha, i^*}$.

For each $\alpha < \lambda$ and $i < \kappa$, let $\dot{C}_{\alpha, i}$ be a canonical name for $C_{\alpha, i}$ (where, if $p \in \mathbb{P}$ and p decides the value of $i(\alpha)$, then $p \Vdash \dot{C}_{\alpha, i} = \emptyset$ for all $i < i(\alpha)$). Find $p \in G$ and a \mathbb{P} -name \dot{D} such that $p \Vdash \dot{D}$ is club in λ and, for all $\alpha \in \dot{D}'$, $\dot{D} \cap \alpha = \dot{C}_{\alpha, i^*}$. Working in V , we play a run of $G_\omega(\mathbb{P})$, $\langle p_n \mid n < \omega \rangle$ with II playing according to the winning strategy described in Proposition 7.2 and I playing to ensure that the following hold:

- $p_1 \leq p$;
- there is a strictly increasing sequence of ordinals $\langle \beta_n \mid n < \omega \rangle$ such that:
 - for all $n < \omega$, $\gamma^{p_{2n}} \leq \beta_n < \gamma^{p_{2n+1}}$ (where we assign $\gamma^{p_0} = 0$);
 - for all $n < \omega$, $p_{2n+1} \Vdash \beta_n \in \dot{D}$.

Let $\gamma = \sup(\{\gamma^{p_n} \mid n < \omega\}) = \sup(\{\beta_n \mid n < \omega\})$. We will define $q \in \mathbb{P}$, a lower bound for $\langle p_n \mid n < \omega \rangle$, of the form $\langle C_{\alpha, i}^q \mid \alpha \leq \gamma, i(\alpha)^q \leq i < \kappa \rangle$. As usual, we only need to specify $i(\gamma)^q$ and $C_{\gamma, i}^q$ for all $i(\gamma)^q \leq i < \kappa$. Let $i(\gamma)^q = i^* + 1$ and, for all $i(\gamma)^q \leq i < \kappa$, let $C_{\gamma, i}^q = \bigcup \{C_{\gamma^{p_{2n}}, i} \mid n < \omega\}$. It is easily verified that q is a lower bound for $\langle p_n \mid n < \omega \rangle$. Since $q \leq p$, $q \Vdash \dot{D}$ is a club," so, as q is a lower bound for $\langle p_n \mid n < \omega \rangle$, $q \Vdash \gamma \in \dot{D}'$. Thus, again because $q \leq p$, $q \Vdash \dot{D} \cap \gamma = \dot{C}_{\gamma, i^*}$. However, as $i(\gamma)^q = i^* + 1$, $q \Vdash \dot{C}_{\gamma, i^*} = \emptyset$, which is a contradiction. \square

8. SEPARATING SQUARES

In [6], we proved that, if λ is the successor of a regular cardinal and $\kappa < \nu < \lambda$ are regular cardinals, it is not necessarily the case that $\square^\kappa(\lambda) \Rightarrow \square^\nu(\lambda)$. In this section, we prove an analogous result when λ is the successor of a singular cardinal. We consider the case $\lambda = \aleph_{\omega+1}$, but the same technique will work for other successors of singular cardinals. We will need the following, a proof of which can be found in [6].

Proposition 8.1. *Let $\kappa < \lambda$ be infinite, regular cardinals, and let $\mathbb{Q} = \mathbb{Q}(\lambda, 1)$ be the forcing to add a $\square(\lambda)$ -sequence. If G is \mathbb{Q} -generic over V , then $\bigcup G$ is a $\square^\kappa(\lambda)$ -sequence.*

Theorem 8.2. *Suppose κ is a supercompact cardinal, GCH holds, and $n < \omega$. Let $\lambda = \kappa^{+\omega+1}$. Then there is a forcing extension in which all cardinals $\leq \aleph_{n+1}$ are preserved, $\kappa = \aleph_{n+2}$, $\lambda = \aleph_{\omega+1}$, $\square^{\aleph_n}(\lambda)$ holds, and $\square^{\aleph_{n+1}}(\lambda)$ fails.*

Proof. The proof follows the proof of Theorem 3.15 in [6]. We thus omit many of the details and refer the reader to the earlier paper.

Let the initial model be called V_0 . In V_0 , let $\mathbb{P} = \text{Coll}(\aleph_{n+1}, < \kappa)$. Let G be \mathbb{P} -generic over V_0 , and let $V = V_0[G]$. Work for now in V . Let $\mathbb{Q} = \mathbb{Q}(\lambda, 1)$, and let

$\dot{\vec{C}}$ be the canonical \mathbb{Q} -name for the $\square(\lambda)$ -sequence added by \mathbb{Q} . Let $\dot{\mathbb{T}}$ be a \mathbb{Q} -name for $\mathbb{T}(\dot{\vec{C}})$.

Working in $V^{\mathbb{Q}}$, we define a sequence of posets $\langle \mathbb{S}_\alpha \mid \alpha \leq \lambda^+ \rangle$ by induction on α . Each \mathbb{S}_α will be λ -distributive and so will preserve the cardinal structure below λ . For each $\beta < \lambda^+$, we will fix an \mathbb{S}_β -name \dot{X}_β for a subset of $S_{\aleph_{n+1}}^\lambda$ such that $\Vdash_{\mathbb{S}_\beta * \mathbb{T}} \text{"}\dot{X}_\beta \text{ is non-stationary.}"$ If $\alpha \leq \lambda^+$, then elements of \mathbb{S}_α are functions s such that:

- (1) $\text{dom}(s) \subseteq \alpha$;
- (2) $|s| \leq \kappa^{+\omega}$;
- (3) for every $\beta \in \text{dom}(s)$, $s(\beta)$ is a closed, bounded subset of λ ;
- (4) for every $\beta \in \text{dom}(s)$, $s \restriction \beta \Vdash_{\mathbb{S}_\beta} \text{"}\dot{X}_\beta \cap s(\beta) = \emptyset\text{"}$.

If $s, t \in \mathbb{S}_\alpha$, then $t \leq s$ iff $\text{dom}(s) \subseteq \text{dom}(t)$ and, for every $\beta \in \text{dom}(s)$, $t(\beta)$ end-extends $s(\beta)$. Let $\mathbb{S} = \mathbb{S}_{\lambda^+}$. It is easily seen that, for every $\alpha < \lambda^+$, $\Vdash_{\mathbb{S}} \text{"}\dot{X}_\alpha \text{ is non-stationary.}"$ Moreover, \mathbb{S} has the λ^+ -chain condition, so, since $2^\lambda = \lambda^+$ in $V^{\mathbb{Q}}$, we can choose the sequence $\langle \dot{X}_\alpha \mid \alpha < \lambda^+ \rangle$ in such a way so that, if \dot{X} is an \mathbb{S} -name for a subset of $S_{\aleph_{n+1}}^\lambda$ and $\Vdash_{\mathbb{S} * \mathbb{T}} \text{"}\dot{X} \text{ is non-stationary,}"$ then already $\Vdash_{\mathbb{S}} \text{"}\dot{X} \text{ is non-stationary.}"$

The following lemmas are proved as in [6].

Lemma 8.3. \mathbb{S} is \aleph_{n+1} -closed.

Lemma 8.4. In V , $\mathbb{Q} * \dot{\mathbb{S}} * \dot{\mathbb{T}}$ has a dense λ -directed closed subset.

$V^{\mathbb{Q} * \dot{\mathbb{S}}}$ will be our desired model. Thus, let H be \mathbb{Q} -generic over V , and let I be \mathbb{S} -generic over $V[H]$. It is clear that, in $V[H * I]$, $\kappa = \aleph_{n+2}$ and $\lambda = \aleph_{\omega+1}$. We first argue that $\square^{\aleph_n}(\lambda)$ holds. Since forcing with \mathbb{Q} adds a $\square^{\aleph_n}(\lambda)$ -sequence, we know that $\square^{\aleph_n}(\lambda)$ holds in $V[H]$. By Proposition 6.4, we can fix a $\square(\lambda)$ -sequence $\vec{C} = \langle C_\alpha \mid \alpha < \lambda \rangle$ such that there is a stationary set $S \subseteq S_{\aleph_n}^\lambda$ such that, for every $\alpha \in S$, $\text{otp}(C_\alpha) = \aleph_n$.

Claim 8.5. In $V[H]$, $S \in I[\lambda]$.

Proof. It is easily seen that $S_\omega^\lambda \in I[\lambda]$ for any regular, uncountable λ , so we may assume that $n > 0$. Let $\vec{a} = \langle a_\alpha \mid \alpha < \lambda \rangle$ be an enumeration of all bounded subsets of λ . Let $E = \{\gamma < \lambda \mid \text{for all } \alpha \leq \beta < \lambda, \text{ there is } \delta < \gamma \text{ such that } C_\beta \cap \alpha = a_\delta\}$. E is a club in λ . We claim that, if $\gamma \in S \cap E$, then γ is approachable with respect to \vec{a} . To see this, fix such a γ . Let $\alpha < \gamma$, and let $\beta = \min(C'_\gamma \setminus \alpha)$. $\beta < \gamma$, $C_\gamma \cap \alpha = C_\beta \cap \alpha$, and there is $\delta < \gamma$ such that $C_\beta \cap \alpha = a_\delta$. Thus, C_γ witnesses that γ is approachable with respect to \vec{a} . \square

Since $S \in I[\lambda]$ and \mathbb{S} is \aleph_{n+1} -closed, Fact 3.6 implies that S remains stationary in $V[H * I]$, so \vec{C} remains a $\square^{\aleph_n}(\lambda)$ -sequence in $V[H * I]$. Note that we don't need to argue separately that there is no thread through \vec{C} in $V[H * I]$, since this is implied by the fact that S is stationary.

We finally show that $\square^{\aleph_{n+1}}(\lambda)$ fails in $V[H * I]$. Suppose on the contrary that, in $V[H * I]$, $\vec{D} = \langle D_\alpha \mid \alpha < \lambda \rangle$ is a $\square(\lambda)$ -sequence and $T \subseteq S_{\aleph_{n+1}}^\lambda$ is stationary such that, for all $\alpha \in T$, $\text{otp}(D_\alpha) < \alpha$. By our construction of \mathbb{S} , we can let J be \mathbb{T} -generic over $V[H * I]$ such that T remains stationary in $V[H * I * J]$. We will reach a contradiction by showing that T cannot remain stationary in $V[H * I * J]$.

In V_0 , let $j : V_0 \rightarrow M_0$ witness that κ is λ^+ -supercompact. By Fact 3.5, $j(\mathbb{P}) = \text{Coll}(\aleph_{n+1}, < j(\kappa)) \cong \mathbb{P} * \dot{\mathbb{Q}} * \dot{\mathbb{S}} * \dot{\mathbb{T}} * \dot{\mathbb{R}}$, where \mathbb{R} is \aleph_{n+1} -closed. Thus, letting K be \mathbb{R} -generic over $V[H * I * J]$, we can lift j to $j : V \rightarrow M_0[G * H * I * J * K] =: M$. In V , let \mathbb{U} be the dense, λ -directed closed subset of $\mathbb{Q} * \dot{\mathbb{S}} * \dot{\mathbb{T}}$. Then $j(\mathbb{U})$ is a dense, $j(\lambda)$ -directed closed subset of $j(\mathbb{Q} * \dot{\mathbb{S}} * \dot{\mathbb{T}})$. Also note that, for every $(q, \dot{s}, \dot{t}) \in H * I * J$, there is $(q', \dot{s}', \dot{t}') \in (H * I * J) \cap \mathbb{U}$ with $(q', \dot{s}', \dot{t}') \leq (q, \dot{s}, \dot{t})$. Then, since M is closed under λ^+ -sequences, $\{j(q, \dot{s}, \dot{t}) \mid (q, \dot{s}, \dot{t}) \in (H * I * J) \cap \mathbb{U}\}$ is a directed subset of $j(\mathbb{U})$ in M of size $\lambda^+ < j(\lambda)$ and thus has a lower bound, $(q^*, \dot{s}^*, \dot{t}^*)$. Then, letting $H^+ * I^+$ be $j(\mathbb{Q} * \dot{\mathbb{S}})$ -generic over $V[H * I * J * K]$ with $(q^*, \dot{s}^*) \in H^+ * I^+$, we can extend j further to $j : V[H * I] \rightarrow M[H^+ * I^+]$.

Let $\eta = \sup(j \text{``}\lambda\text{'}) < \lambda$. Note that, in $V[H * I * J * K * H^+ * I^+]$, $\text{cf}(\eta) = \aleph_{n+1}$ and $j \text{``}\lambda$ is $(< \kappa)$ -club in η . Let $j(\vec{D}) = \vec{E} = \langle E_\alpha \mid \alpha < j(\lambda) \rangle$. Let $F = \{\alpha \in S_{<\kappa}^\lambda \mid j(\alpha) \in E'_\eta\}$. F is $(< \kappa)$ -club in λ . If $\alpha < \beta$ are both in F , then $j(\alpha), j(\beta) \in E'_\eta$, so $j(D_\alpha) = E_{j(\alpha)} = E_\eta \cap j(\alpha)$ and $j(D_\beta) = E_{j(\beta)} = E_\eta \cap j(\beta)$, so $j(D_\alpha) = j(D_\beta) \cap j(\alpha)$. By elementarity, $D_\alpha = D_\beta \cap \alpha$, so $F^* = \bigcup_{\alpha \in F} D_\alpha$ is a thread through \vec{D} . $F^* \in V[H * I * J * K * H^+ * I^+]$. Since $F^* \subseteq \lambda$ and $H^+ * I^+$ is generic for λ^+ -distributive forcing, $F^* \in V[H * I * J * K]$.

The proof of the following claim is standard. We include it for completeness.

Claim 8.6. *There is a thread through \vec{D} in $V[H * I * J]$.*

Proof. Suppose not. Work in $V[H * I * J]$. Let \dot{F} be an \mathbb{R} -name for F^* , and let $r \in \mathbb{R}$ force that \dot{F} is a thread through \vec{D} . Since there is no thread through \vec{D} in $V[H * I * J]$, we can find $r^0, r^1 \leq r$ and $\alpha < \lambda$ such that $r^0 \Vdash \text{``}\alpha \in \dot{F}\text{''}$ and $r^1 \Vdash \text{``}\alpha \notin \dot{F}\text{''}$. Now, inductively define conditions $\langle r_m^i \mid i < 2, m < \omega \rangle$ and ordinals $\langle \beta_m^i \mid i < 2, m < \omega \rangle$ such that:

- for each $i < 2$, $\langle r_m^i \mid m < \omega \rangle$ is a decreasing sequence of conditions from \mathbb{R} and $r_0^i \leq r^i$;
- for all $m < \omega$, $\beta_m^0 < \beta_m^1 < \beta_{m+1}^0 < \lambda$;
- for all $i < 2$ and $m < \omega$, $r_m^i \Vdash \text{``}\beta_m^i \in \dot{F}\text{''}$.

Let $\gamma = \sup(\{\beta_m^0 \mid m < \omega\}) = \sup(\{\beta_m^1 \mid m < \omega\})$ and, for $i < 2$, use the closure of \mathbb{R} to find a lower bound s_i for $\langle r_m^i \mid m < \omega \rangle$. For each $i < 2$, $s_i \Vdash \text{``}\gamma \in \lim(\dot{F})\text{''}$, so $s_i \Vdash \text{``}\dot{F} \cap \gamma = C_\gamma\text{''}$. Then $s_0 \leq r^0$ implies that $\alpha \in C_\gamma$, while $s_1 \leq r^1$ implies $\alpha \notin C_\gamma$. Contradiction. \square

Let A be a thread through \vec{D} in $V[H * I * J]$. Then $A^* = \{\alpha \in A \mid \text{otp}(A \cap \alpha) = \alpha\}$ is club in λ and $A^* \cap T = \emptyset$, contradicting the fact that T is stationary in $V[H * I * J]$. \square

9. DERIVED SYSTEMS

One of the most useful properties of systems is that, when \mathbb{P} is a sufficiently small forcing poset, a \mathbb{P} -name for a system gives rise to a system in the ground model. Let \mathbb{P} be a forcing poset, let $\tau < \lambda$ be cardinals, with λ regular and $|\mathbb{P}| < \lambda$, and suppose $\dot{S} = \langle \{\{\alpha\} \times \dot{\kappa}_\alpha \mid \alpha \in \dot{I}\}, \{\dot{R}_i \mid i < \tau\} \rangle$ is a \mathbb{P} -name for a λ -system. Since every λ -system $\langle \{\{\alpha\} \times \kappa_\alpha \mid \alpha \in I\}, \mathcal{R} \rangle$ is isomorphic to one in which $I = \lambda$, we may assume $\dot{I} = \check{\lambda}$. For each $\alpha < \lambda$, find $p_\alpha \in \mathbb{P}$ deciding the value of $\dot{\kappa}_\alpha$. Since $|\mathbb{P}| < \lambda$, there is a $p \in \mathbb{P}$ such that, for unboundedly many $\alpha < \lambda$, $p_\alpha = p$. Thus, by passing to a name for a subsystem and working below a condition in \mathbb{P} , we may

assume \dot{S} is of the form $\langle \{\{\alpha\} \times \kappa_\alpha \mid \alpha < \lambda\}, \{\dot{R}_i \mid i < \tau\} \rangle$. In V , we define the derived system $D_{\mathbb{P}}(\dot{S}) = \langle \{\{\alpha\} \times \kappa_\alpha \mid \alpha < \lambda\}, \{R_{i,p} \mid i < \tau, p \in \mathbb{P}\} \rangle$ by letting, for every $\alpha_0 < \alpha_1 < \lambda$, $\beta_0 < \kappa_{\alpha_0}$, $\beta_1 < \kappa_{\alpha_1}$, $i < \tau$, and $p \in \mathbb{P}$, $(\alpha_0, \beta_0) <_{R_{i,p}} (\alpha_1, \beta_1)$ iff $p \Vdash “(\alpha_0, \beta_0) <_{\dot{R}_i} (\alpha_1, \beta_1)”$. It is easily verified that $D_{\mathbb{P}}(\dot{S})$ is a λ -system and $\text{width}(D_{\mathbb{P}}(\dot{S})) = \max(\{\sup(\{\kappa_\alpha \mid \alpha < \lambda\}), \tau, |\mathbb{P}|\})$.

Proposition 9.1. *Suppose \mathbb{P} is a forcing poset, $\tau < \lambda$ are cardinals, with λ regular, and $\dot{S} = \langle \{\{\alpha\} \times \kappa_\alpha \mid \alpha < \lambda\}, \{\dot{R}_i \mid i < \tau\} \rangle$ is a \mathbb{P} -name for a λ -system. If $\Vdash_{\mathbb{P}} “\dot{S}$ has no cofinal branch,” then $D_{\mathbb{P}}(\dot{S})$ has no cofinal branch in V*

Proof. Suppose $i < \tau$, $p \in \mathbb{P}$, and b is a cofinal branch through $R_{i,p}$ in $D_{\mathbb{P}}(\dot{S})$. Then $p \Vdash “b$ is a cofinal branch through \dot{R}_i in $\dot{S}.”$ \square

Proposition 9.2. *If $\mu \leq \lambda$, with λ regular, then $NSP(\mu, \lambda)$ is indestructible under forcing with posets \mathbb{P} such that $|\mathbb{P}| < \mu$ and $|\mathbb{P}|^+ < \lambda$.*

Proof. Suppose that $\mu \leq \lambda$, $NSP(\mu, \lambda)$ holds in V , \mathbb{P} is a forcing poset, $|\mathbb{P}| < \mu$, and $|\mathbb{P}|^+ < \lambda$. Suppose for sake of contradiction that there is $p \in \mathbb{P}$ and a \mathbb{P} -name $\dot{S} = \langle \lambda \times \kappa, \{\dot{R}_i \mid i < \tau\} \rangle$ such that $\kappa, \tau < \mu$ and $p \Vdash_{\mathbb{P}} “\dot{S}$ is a narrow λ -system with no cofinal branch.” Let $\mathbb{Q} = \mathbb{P}/p = \{q \in \mathbb{P} \mid q \leq p\}$. Then, re-interpreting \dot{S} as a \mathbb{Q} -name, we obtain $\Vdash_{\mathbb{Q}} “\dot{S}$ is a narrow λ -system with no cofinal branch.” Then, in V , $D_{\mathbb{Q}}(\dot{S})$ is a λ -system with no cofinal branch, $\text{width}(D_{\mathbb{Q}}(\dot{S})) = \max(\{\kappa, \tau, |\mathbb{P}|\}) < \mu$, and $\text{width}(D_{\mathbb{Q}}(\dot{S}))^+ < \lambda$, contradicting $NSP(\mu, \lambda)$. \square

Proposition 9.3. *Let κ be an infinite cardinal. There is a κ^+ -system of width κ with no cofinal branch.*

Proof. Let $\mathbb{P} = \text{Coll}(\omega, \kappa)$. Then $|\mathbb{P}| = \kappa$ and, in $V^{\mathbb{P}}$, $\kappa^+ = \omega_1$. Thus, in $V^{\mathbb{P}}$, there is a κ^+ -Aronszajn tree, T . We may assume that, for all $\alpha < \kappa^+$, level α of T is the set $\{\alpha\} \times \omega$. T can thus be thought of as a κ^+ -system of width ω with one relation (the tree relation). Letting \dot{T} be a \mathbb{P} -name for T , we may form the derived system $D_{\mathbb{P}}(\dot{T})$ in V . $D_{\mathbb{P}}(\dot{T})$ is a κ^+ -system of width $|\mathbb{P}| = \kappa$. Since $\Vdash_{\mathbb{P}} “\dot{T}$ has no cofinal branch,” $D_{\mathbb{P}}(\dot{T})$ has no cofinal branch in V . \square

We now introduce another variation on Jensen’s square principle.

Definition Suppose $\kappa \leq \mu$ are infinite cardinals, with κ regular. $\langle C_\alpha \mid \alpha \in A \rangle$ is a $\square_{\mu}^{\geq \kappa}$ -sequence if:

- (1) $S_{\geq \kappa}^{\mu^+} \subseteq A \subseteq \lim(\mu^+)$.
- (2) For all $\alpha \in A$, C_α is club in α and $\text{otp}(C_\alpha) \leq \mu$.
- (3) For all $\beta \in A$, if $\alpha \in C'_\beta$, then $\alpha \in A$ and $C_\beta \cap \alpha = C_\alpha$.

$\square_{\mu}^{\geq \kappa}$ is the assertion that a $\square_{\mu}^{\geq \kappa}$ -sequence exists.

Square principles of this sort were first studied by Baumgartner, in unpublished work. Let $\mathbb{B}(\mu, \kappa)$ be the forcing poset whose conditions are of the form $p = \langle C_\alpha^p \mid \alpha \in s^p \rangle$ such that:

- s^p is a bounded subset of μ^+ with a maximal element, γ^p .
- $\gamma^p \cap \text{cof}(\geq \kappa) \subseteq s^p$.
- For all $\alpha \in s^p$, C_α^p is a club in α and $\text{otp}(C_\alpha) \leq \mu$.
- For all $\beta \in s^p$, if $\alpha \in (C_\beta^p)'$, then $\alpha \in s^p$ and $C_\beta^p \cap \alpha = C_\alpha^p$.

If $p, q \in \mathbb{B}(\mu, \kappa)$, then $q \leq p$ if s^q end-extends s^p and, for all $\alpha \in s^p$, $C_\alpha^q = C_\alpha^p$. The following is easily proven in the usual manner (see [1] for details).

Proposition 9.4. *Let $\kappa \leq \mu$ be infinite cardinals, with κ regular.*

- (1) $\mathbb{B}(\mu, \kappa)$ is κ -directed closed.
- (2) $\mathbb{B}(\mu, \kappa)$ is $\mu + 1$ -strategically closed.
- (3) $\Vdash_{\mathbb{B}(\mu, \kappa)} \text{"}\square_{\mu}^{\geq \kappa} \text{ holds."}$

Proposition 9.5. *Let $\kappa < \mu$ be infinite cardinals, and suppose $\square_{\mu}^{\geq \kappa^+}$ holds. Then there is a μ^+ -system of width κ with no cofinal branch.*

Proof. Let $\mathbb{P} = \text{Coll}(\omega, \kappa)$.

Claim 9.6. *In $V^{\mathbb{P}}$, \square_{μ} holds.*

Proof. In V , let $\vec{C} = \langle C_{\alpha} \mid \alpha \in A \rangle$ be a $\square_{\mu}^{\geq \kappa^+}$ -sequence. $S_{\geq \kappa^+}^{\mu^+} \subseteq A$, so, in $V^{\mathbb{P}}$, if $\alpha \in \lim(\mu^+) \setminus A$, then $\text{cf}(\alpha) = \omega$. Thus, in $V^{\mathbb{P}}$, we can define a \square_{μ} -sequence $\vec{D} = \langle D_{\alpha} \mid \alpha < \mu^+ \rangle$ by letting $D_{\alpha} = C_{\alpha}$ for all $\alpha \in A$ and, for all $\alpha \in \lim(\mu^+) \setminus A$, letting D_{α} be an arbitrary ω -sequence cofinal in α . \square

Therefore, by Proposition 6.6, there is, in $V^{\mathbb{P}}$, a μ^+ -system S of width ω with no cofinal branch. Let \dot{S} be a name for such an S . Then the derived system $D_{\mathbb{P}}(\dot{S})$ is, in V , a μ^+ -system of width κ with no cofinal branch. \square

We can use this to show that, for example, the narrow system property at $\aleph_{\omega+1}$ can hold for narrow systems of arbitrarily high width below \aleph_{ω} while failing in general.

Corollary 9.7. *Suppose λ is a supercompact cardinal and $n < \omega$. Then there is a forcing extension in which every narrow $\aleph_{\omega+1}$ -system of width $\leq \aleph_n$ has a cofinal branch but there is a narrow $\aleph_{\omega+1}$ -system of width \aleph_{n+1} with no cofinal branch.*

Proof. Let $\mathbb{P} = \text{Coll}(\aleph_{n+1}, < \lambda)$, and let G be \mathbb{P} -generic over V . In $V[G]$, let $\mathbb{Q} = \mathbb{B}(\lambda^{+\omega}, \lambda)$, and let H be \mathbb{Q} -generic over $V[G]$. Note that, in $V[G * H]$, $\lambda = \aleph_{n+2}$ and $\lambda^{+\omega+1} = \aleph_{\omega+1}$. Since, in $V[G]$, \mathbb{Q} is λ -directed closed, Theorem 5.1 implies that, in $V[G * H]$, $NSP(\aleph_{n+1}, \geq \aleph_{n+2})$ holds. In particular, every narrow $\aleph_{\omega+1}$ -system of width $\leq \aleph_n$ has a cofinal branch. On the other hand, $\square_{\aleph_{\omega}}^{\geq \aleph_{n+2}}$ holds in $V[G * H]$, so, by 9.5, there is a narrow $\aleph_{\omega+1}$ -system of width \aleph_{n+1} with no cofinal branch. \square

10. THE PROPER FORCING AXIOM AND NARROW SYSTEMS

In this section, we investigate the extent to which the Proper Forcing Axiom (PFA) influences narrow systems. We first recall the notion of a guessing model, introduced by Viale and Weiss in [13].

Definition Let θ be a regular cardinal, and let $M \prec H(\theta)$.

- (1) Suppose $X \in M$ and $d \subseteq X$.
 - (a) d is M -approximated if, for every countable $z \in M$, $z \cap d \in M$.
 - (b) d is M -guessed if there is $e \in M$ such that $e \cap M = d \cap M$.
- (2) M is a *guessing model* if every M -approximated d is M -guessed.
- (3) If $\kappa \leq \theta$, then $\mathcal{G}_{\kappa}(H(\theta)) = \{M \prec H(\theta) \mid |M| < \kappa \text{ and } M \text{ is a guessing model}\}$.

The following is proven in [13].

Theorem 10.1. *Suppose PFA holds. Then $\mathcal{G}_{\omega_2}(H(\theta))$ is stationary in $\mathcal{P}_{\omega_2}(H(\theta))$ for every regular $\theta \geq \omega_2$.*

We use this to prove the following result.

Theorem 10.2. *Suppose PFA holds. Then $NSP(\omega_1, \geq \omega_2)$ holds.*

Proof. Suppose $\lambda \geq \omega_2$ is a regular cardinal and $S = \langle I \times \omega, \mathcal{R} \rangle$ is a λ -system, with $|\mathcal{R}| \leq \omega$. As before, we may assume $I = \lambda$. We must produce a cofinal branch through S .

Let θ be a sufficiently large regular cardinal, and let $M \in \mathcal{G}_{\omega_2}(H(\theta))$ be such that $S \in M$. Let $\delta = \sup(M \cap \lambda)$.

Claim 10.3. $\text{cf}(\delta) > \omega$.

Proof. Suppose for sake of contradiction that $\text{cf}(\delta) = \omega$. Let $A = \{\alpha_n \mid n < \omega\}$ be such that $A \subseteq M$, A is cofinal in δ , and, for all $n < \omega$, $\alpha_n < \alpha_{n+1}$. Let $z \in M$ be a countable set. Then $M \models$ “ $z \cap \lambda$ is bounded below λ ”, so z is bounded below δ . This implies that $z \cap A$ is a finite set and hence a member of M . Thus, A is M -approximated, so, since M is a guessing model, there is $B \in M$ such that $B \cap M = A \cap M = A$. Moreover, B is countable, as otherwise $B \cap M$ would be uncountable. But then $\delta \leq \sup(B \cap \lambda) < \lambda$ and $\sup(B \cap \lambda) \in M$, contradicting $\delta = \sup(M \cap \lambda)$. \square

For each $n < \omega$ and $R \in \mathcal{R}$, let $d_{n,R} = \{u \in S \cap M \mid u <_R (\delta, n)\}$. $d_{n,R}$ is then a branch of S through R . For every $\alpha \in M \cap \delta$, there are $n < \omega$ and $R \in \mathcal{R}$ such that $d_{n,R} \cap S_\alpha \neq \emptyset$. Thus, since $\text{cf}(\delta) > \omega$, there are $n^* < \omega$ and $R^* \in \mathcal{R}$ such that $d_{n^*,R^*} \cap S_\alpha \neq \emptyset$ for cofinally many $\alpha \in M \cap \delta$. Fix such an n^* and R^* , and let $d = d_{n^*,R^*}$.

Claim 10.4. d is M -approximated.

Proof. Suppose $z \in M$ is countable. Then $a_z = \{\alpha \mid d \cap z \cap S_\alpha\}$ is bounded below δ . Let $\beta \in M \cap \delta$ be such that $a_z \subseteq \beta$ and $d \cap S_\beta \neq \emptyset$. Let $d \cap S_\beta = \{v\}$. Then $z \cap d = \{u \in S \cap z \mid u <_{R^*} v\}$. Everything used to define this set is in M , so $z \cap d \in M$. \square

Since M is a guessing model and d is M -approximated, there is $b \in M$ such that $b \cap M = d \cap M$. But then $M \models$ “ b is a cofinal branch through S ,” so, by elementarity, b is in fact a cofinal branch through S . \square

The following result shows that Theorem 10.2 is sharp.

Theorem 10.5. *PFA does not imply $NSP(\omega_2, \mu^+)$ for any $\mu \geq \omega_2$.*

Proof. Suppose κ is supercompact, and let $\mu \geq \kappa$. Assume that the supercompactness of κ is indestructible under κ -directed closed forcing. Let $\mathbb{B} = \mathbb{B}(\mu, \kappa)$ be the κ -directed closed forcing to add a $\square_\mu^{\geq \kappa}$ -sequence. In $V^{\mathbb{B}}$, κ remains supercompact, so there is a poset \mathbb{P} such that \mathbb{P} preserves all cardinals $\geq \kappa$ and $\Vdash_{\mathbb{P}}$ “PFA and $\kappa = \aleph_2$ ”. Then, in $V^{\mathbb{B} * \mathbb{P}}$, PFA holds, but also $\square_\mu^{\geq \aleph_2}$ holds, which, by Proposition 9.5, implies the existence of a μ^+ -system of width \aleph_1 with no cofinal branch and hence the failure of $NSP(\omega_2, \mu^+)$. \square

11. OPEN QUESTIONS

In the final section we collect a few as-yet-unanswered questions about narrow systems.

Question Is it consistent that there is a singular cardinal μ such that the tree property holds at μ^+ but $NSP(\mu^+)$ fails?

In all known models for the tree property at the successor of a singular cardinal, the narrow system property holds and is in fact a key component in the verification of the tree property, so a positive answer to Question 11 would seem to require some new ideas.

Question Suppose μ is a singular cardinal and \square_μ^* holds. Must there be a μ^+ -tree that does not admit a narrow system?

First note that a cofinal branch through a μ^+ tree easily yields a narrow system, so a μ^+ -tree that does not admit a narrow system must be a μ^+ -Aronszajn tree. Also, as mentioned before, in models in which \square_μ^* and $NSP(\mu^+)$ both hold, there are μ^+ -Aronszajn trees and all such trees do not admit narrow systems. Also, note that, if $\text{cf}(\mu) < \kappa < \mu$ and κ is strongly compact, then every μ^+ -tree admits a narrow system.

Consideration of the following questions about the tree property led to the results in this paper. They remain unanswered.

Question Is it consistent that the tree property holds at $\aleph_{\omega+1}$ and every stationary subset of $\aleph_{\omega+1}$ reflects?

If \aleph_ω is strong limit and every stationary subset of $\aleph_{\omega+1}$ reflects, then AP_{\aleph_ω} holds. Thus, the following may be relevant in answering Question 11.

Question Is it consistent that there is a singular cardinal μ such that AP_μ and the tree property at μ^+ hold simultaneously?

We do not even know the situation in the following seemingly simple model.

Question Suppose $\langle \kappa_n \mid n < \omega \rangle$ is an increasing sequence of supercompact cardinals and $\mu = \sup(\{\kappa_n \mid n < \omega\})$. Let \mathbb{P} be the forcing poset to shoot a club in μ^+ through the set of approachable points. In $V^\mathbb{P}$, does the tree property hold at μ^+ ?

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